The Risk Channel of Unconventional Monetary Policy*

Dejanir H. Silva†

August, 2016
[latest version]

Abstract

This paper examines how unconventional monetary policy affects asset prices and macroeconomic conditions by reallocating risk in the economy. I consider an environment with two main ingredients: heterogeneity in risk tolerance and limited asset market participation. Risk-tolerant investors take leveraged positions, exposing the economy to balance sheet recessions. Limited asset market participation implies the balance sheet of the central bank is non-neutral. Unconventional monetary policy reduces the risk premium and endogenous volatility. During balance sheet recessions, asset purchases boost investment and growth. In contrast, during normal times, the expectation of future interventions reduces growth by its impact on savings. A commitment by the central bank to unwind its portfolio early, conditional on the recovery of leveraged institutions' balance sheet, reduces the risk premium by more than strategies involving holding its portfolio for longer. Asset purchases also have implications for the concentration of risk. Leveraged institutions respond to the policy by reducing risk-taking relatively more than risk-averse investors. As risk concentration falls, the probability of negative tail-events is reduced, enhancing financial stability.

JEL classification: E44, E58, G10

*This paper is a revised version of the first chapter of my PhD dissertation at MIT. I am extremely grateful to my advisors Robert Townsend, Iván Werning, Alp Simsek, and Leonid Kogan for invaluable guidance and support. I also thank Rodrigo Adão, George-Marios Angeletos, Nicolas Caramp, Ricardo Caballero, Victor Duarte, Ernest Liu, Juan Passadore, Matthew Rognlie, Andres Sarto, Ludwig Straub. I thank the Macro Financial Modeling (MFM) group for financial support.

†University of Illinois, Urbana-Champaing. Email: dejanir@illinois.edu

1 Introduction

Unconventional monetary policy has been at the center of policy debate since the onset of the Great Recession. The policy consisted of large-scale asset purchases (LSAPs) in an attempt to compress risk premium and ease credit conditions.[1] Given the unprecedented character of such policies, the transmission mechanism of LSAPs as well as the potential side-effects of these policies remains a source of debate. What is the effect of different "exit strategies"? Does unconventional monetary policy induce more risk-taking in the financial sector? Can the expectation of future interventions affect the economy even after the central bank unwind its portfolio? In this paper, I provide a framework for the analysis of the transmission mechanism of LSAPs which allow us to address these questions.

I propose a *risk channel* of unconventional monetary policy. This channel operates through changes in the supply of risk to marginal investors, resulting in changes in risk premium and ultimately affecting risk-taking and economic growth. Key for my results are two forms of heterogeneity: differences in market access and, among market participants, differences in risk tolerance. Limited asset market participation is important to guarantee that changes in the central bank balance sheet have real effects. Heterogeneity in risk tolerance will lead to a *countercyclical aggregate risk aversion*, exposing the economy to *balance sheet recessions*, i.e., a drop in asset prices and growth associated with a weak balance sheet of (risk-tolerant) financial intermediaries. In this context, LSAPs can be used to counteract the effects of balance sheet recessions, with consequences to portfolio and savings decisions of investors.

The environment consists of a stochastic growth model in continuous-time with heterogeneous agents. The economy is subject to limited asset market participation and market participants have heterogeneous preferences. Market participants (investors) can trade without any frictions while non-participants simply consume their endowment plus any transfers from the government. A group of relatively risk-tolerant investors (financial intermediaries) obtain short-term funding from a group of more risk-averse investors (savers) to finance risky investments.[2] The central bank invests in risky and riskless assets (reserves) and rebates the proceeds of the investment to non-participants according to given policy rules. In order to isolate the role of the risk channel, I abstract from features present in other theories of LSAPs, like liquidity frictions in the financial sector, limited commitment by the central bank, or a special role for the central bank's liability.[3] Importantly, even in the absence of these features, asset purchases by the central bank can have real effects.

In the first part of the paper, I show the laissez-faire economy is subject to balance sheet recessions. The main feature driving this result is countercyclical aggregate risk aversion. Given the high demand for safe assets from risk-averse savers, financial intermediaries issue riskless assets and invest in risky

[1]See Fawley and Neely (2013) for a detailed discussion of large-scale asset purchases by the Federal Reserve, its motivation, and the comparison with the experience of other central banks. LSAPs is typically referred to as *quantitative easing*.

[2]Intermediaries rely on short-term borrowing to finance risky investment. They should be interpret as leveraged institutions in general, like commercial bank, investment banks, and hedge funds. Savers correspond generically to funding institutions. Gertler et al. (2015) in a related setting focus instead on the distinction between retail and wholesale banks.

[3]I discuss the alternative theories in more detail in the literature review at end of this section.

assets. Financial intermediaries will expose themselves to *risk-mismatch* as their assets are riskier than their liabilities. After a negative shock, their share of wealth will fall, increasing the average (wealth-weighted) risk aversion. A countercyclical aggregate risk aversion implies that risk premium rises and the interest rate falls after a negative shock. The increase in risk premium will depress asset prices and ultimately will reduce investment and growth. Heterogeneity in risk-tolerance is key for this result. Under homogeneous preferences, intermediaries and savers would choose the same exposure to risk and their relative wealth position would not respond to shocks. The economy would jump to a *balance growth path* with no variation in returns or investment. Brunnermeier and Sannikov (2014) and He and Krishnamurthy (2012) obtain balance sheet recessions in a setting with homogeneous preferences by limiting trade in aggregate risk. Di Tella (2012) allows for aggregate risk sharing, but a balance sheet channel only arises in the presence of idiosyncratic uncertainty shocks and moral hazard frictions. Preference heterogeneity allows me to keep the tractability of Brownian investment shocks without imposing any restriction on the ability of investors to trade aggregate risk.

The assumption of limited asset market participation is crucial to guarantee LSAPs can affect asset prices. Under full participation, a result akin to Modigliani-Miller/Ricardian Equivalence holds, and investors will exactly offset any changes in asset holdings of the central bank.[4] The assumption of limited participation captures the fact that the central bank intervened in relatively sophisticated markets, like MBSs, which are not readily accessible to everyone. As the central bank rebates the proceeds from its investment to workers, less risk will be held by the marginal investors, affecting asset prices.

I follow the tradition in the analysis of conventional monetary policy, and I specify the policy instruments of the central bank by *policy rules*. In particular, the portfolio of the central bank is a function of the balance sheet position of financial intermediaries. I focus a rule where central bank intervenes only when financial intermediaries' balance sheet is sufficiently weak, i.e., when aggregate risk aversion is high and asset prices are depressed.[5] This corresponds to an *unconventional Greenspan's Put*, where instead of easing conventional monetary policy when asset prices are low, the central bank expands its balance sheet in states where asset prices are depressed. Given the specification of the policy rule, I solve the model numerically and calibrate it to US data. To capture the nonlinearities involved in movements in risk premium and in risk-taking of investors, it is important to use global solution methods instead of local approximations around a steady state. One advantage of the continuous-time setting is to allow for an effective solution method. The equilibrium can be obtained as the solution to a system of partial differential equations (PDEs) in the two state variables: the share of wealth of intermediaries and of the central bank.

Asset purchases by the central bank reduce the risk premium. LSAPs reduce the net supply of risk to investors, so in equilibrium a smaller return per unit of risk (Sharpe ratio) is required to clear the

[4] Wallace (1981) was the first to derive such neutrality result. Eggertsson and Woodford (2003) derives a similar result in an economy with sticky prices.

[5] In parallel work, Silva (2015), I discuss the determination of optimal policy in a two-period setting. I introduce a moral hazard friction in the financial sector, and find the optimal policy would balance the pecuniary externality with the limited participation problem. Optimal policy would limit variations in price, similar to the effects of the policy proposed here.

market. Endogenous volatility also fall in response to policy, so risk premium falls. Interest rates rise due to a combination of two effects. First, as the purchase of risky assets is financed by an increase in reserves, a higher interest rate is required to induce investors to hold the larger supply of safe assets. Second, as the intervention reduces volatility, investors have less of a precautionary savings motive. Importantly, this effect is present even when the central bank is not currently intervening. The expectation of intervention during crises reduce savings in normal times. In equilibrium, investment will fall to accommodate higher consumption. Hence, expectation of future intervention will reduce economic growth in normal times.

One concern with unconventional monetary policy is that it could create financial stability risks. In contrast, I find no support for such concerns. I identify two effects of LSAPs on the concentration of risk in the hands of financial intermediaries. First, a *hegding effect*. Since returns are countercyclical, intermediaries tilt their portfolio toward states with high returns, i.e., they hedge variations in returns. After the intervention returns are less countercyclical. In response, intermediaries tend to increase exposure to risk, given the weaker incentive to hedge. This argument is in line with the concerns of critics. However, I identify a second effect, a *return sensitivity effect*. Since intermediaries are more risk tolerant, they are also more sensitive to returns. As the central bank compress returns, intermediaries are the ones with the stronger incentive to sell assets to the central bank, reducing their exposure to risk relative to savers. I find the return sensitivity effect dominates, reducing the concentration of risk in equilibrium. Endogenous volatility is related to the concentration of risk. LSAPs will reduce endogenous volatility and, in the stationary distribution, the probability of large drops in asset prices and growth.

Another source of debate was the role of different exit strategies. I capture the effect of (state-contingent) exit strategies by comparing two policy rules that are identical in states where the balance sheet of intermediaries are weak, but the policies differ as the economy recovers. Perhaps surprisingly, I find that the policy rule where the central bank sells more aggressively as the economy recovers, the "early-exit", *amplifies* the effects of LSAPs during crises. The intuition for this result is that returns are less countercyclical under the early-exit policy, inducing intermediaries to take more risk. Given the higher demand for risk, the market price of risk falls by more under early-exit.

I consider two additional issues: the effectiveness of LSAPs and its impact on the term premium. I find the marginal effect of the policy is higher when intermediaries have a weak balance sheet. The reason is that in those states savers have a greater weight in the demand for risk. Since savers are relatively insensitive to returns, the risk premium falls by more to induce them to sell their assets. The marginal effect increases with the size of the intervention. This suggests that measuring the effect of the policy when the intervention is still small will not capture the impact of the policy after it is scaled up. I also consider the effect on long-term bonds. Since bonds increase in value after a negative shock, they act as an insurance. Since LSAPs reduce endogenous volatility, the demand for insurance falls, rising the term premium and reducing the price of bonds. Hence, the policy will have a differential effect on assets depending on the relative importance of term and risk premium to price that asset.

Literature review. This paper is related to several strands of the literature in macroeconomics and finance. A large literature evaluates empirically LSAPs (Gagnon et al. (2011), Krishnamurthy and Vissing-Jorgensen (2011), D'Amico and King (2013), Greenwood and Vayanos (2014), see Joyce et al. (2012) for a survey). The main conclusion of these studies is that unconventional measures affect asset prices, even though there is a debate about the precise mechanism. Several channels have been proposed in the literature, e.g. a *liquidity channel* (Gertler and Kiyotaki (2010), Del Negro et al. (2011), Curdia and Woodford (2011); Gertler and Karadi (2011, 2013), Williamson (2012), Araújo et al. (2015)), a *signalling channel* (Bhatarai et al. (2014), Berriel and Mendes (2015)), an *asset scarcity/safety premium channel* (Krishnamurthy and Vissing-Jorgensen (2012), Caballero and Farhi (2014)).[6] The liquidity channel emphasizes the role of the central bank in performing intermediation when banks are liquidity constrained. The signaling channel corresponds to the effect of changes in the portfolio of the central bank on the expectation of the path of future interest rates. Asset scarcity theories emphasize the special role of the central bank liability as a safe asset. My research focus on a *risk channel* of unconventional monetary policy and shows how the balance sheet of the central bank have real effects even if banks are unconstrained, the central bank has full commitment, and its liability plays no special role.

This paper is related to the literature on the macroeconomic effects of shocks to balance sheets of firms and banks (e.g. Holmstrom and Tirole (1997), Bernanke et al. (1999), Adrian et al. (2010), He and Krishnamurthy (2013), Brunnermeier and Sannikov (2014)). In contrast to this literature, I assume intermediaries can trade aggregate risk without frictions. Di Tella (2012) adopts the same assumption, but balance sheet recessions arise by a combination of idiosyncratic uncertainty shocks and strong income effects on portfolio choice. In his setting, experts tilt their portfolio toward states with *low* returns, as those states require more resources to achieve a given utility level (an income effect). Since idiosyncratic returns decrease after a positive shock, experts have an incentive to take more risk, explaining why risk is concentrated in his model. In my setting, balance sheet recessions arise due to differences in risk aversion, regardless of whether income or substitution effects dominate on portfolio choice.

A recent literature has considered the impact of monetary policy on financial stability and intermediaries' risk-taking. Diamond and Rajan (2012) and Brunnermeier and Sannikov (2015) illustrate how the expectation of central bank intervention can induce banks to take more risk. I show that unconventional monetary actually *reduces* the concentration of risks in intermediaries, despite the "stealth recapitalization" of banks generated by this policy.

An old literature emphasized the role of *portfolio balance effects* (Gurley et al. (1960),Tobin and Brainard (1963), Tobin (1969), Brunner and Meltzer (1973)). The key assumption of this literature was that assets are imperfect substitutes (usually for unmodelled reasons), so the central bank can affect the return of different assets by affecting their relative supply. I explore a similar mechanism, with risk

[6] A related literature looks at how quantitative easing can potentially affect the solvency of the central bank (see Hall and Reis (2013), Reis (2015)).

being the source of differentiation, and emphasize that limited participation is needed in addition to imperfect substitutability.[7]

My analysis is also related to the literature on the effects of limited asset market participation. There is a long tradition of models of limited participation to study conventional monetary policy (Grossman and Weiss (1983), Rotemberg (1984), Alvarez et al. (2002)). I follow this tradition and study unconventional monetary policy by considering a tractable form of limited participation. A different literature focused on the effects of limited participation on risk premia and volatility (Mankiw and Zeldes (1991),Allen and Gale (1994), Basak and Cuoco (1998),Brav et al. (2002), Guvenen (2009)). Limited participation plays a different role in my analysis than in this literature, as it allows for real effects of changes in the central bank balance sheet, instead of acting as factor contributing to the concentration of risk on market participants.

A different form of limited participation appears in preferred habitat models. Effects of bond purchases by the central bank are usually interpreted using such theories. Vayanos and Vila (2009) provide a formalization in a setting with risk-averse arbitrageurs and investors that invest only in specific maturities of bonds and explore the implications for bond premia. In contrast, I emphasize that bond purchases can affect bond prices even if market participants trade in all maturities. Chien et al. (2012) provide another example of limited access to financial markets, where a set of investors does not rebalance their portfolio frequently.

Methodologically, my research draws on the asset pricing literature on heterogeneous investors (Constantinides (1982), Dumas (1989), Wang (1996), Chan and Kogan (2002), Gomes and Michaelides (2005), Gârleanu and Pedersen (2011), Longstaff and Wang (2012), Gârleanu and Panageas (2015)). My work is close to the papers by Drechsler et al. (2014) and Barro and Mollerus (2014). Both papers adopt Epstein-Zin preferences and follow Longstaff and Wang (2012) in modeling intermediaries as (relatively) risk-tolerant agents. Barro and Mollerus (2014) study the creation of safe assets but abstracts from limited participation, so the government's balance sheet position is neutral. Drechsler et al. (2014) study conventional monetary policy by assuming a special role for the central bank liability. I assume perfect substitutability between intermediaries' and central bank's liability and focus instead on unconventional monetary policy. [8]

Layout. The remainder of the paper is organized as follows. Section 2 presents the baseline model, and section 3 describes the characterization of the equilibrium. Section 4 discuss the effects of LSAPs during a balance sheet recession. Section 5 shows the impact of policy intervention to financial stability and section 6 discuss exit strategies. Section 7 presents the extensions: a discussion of the effectiveness of unconventional monetary policy, and the effects of policy on the term premium. Section 8 concludes.

[7]See Andrés et al. (2004) and Chen et al. (2012) for attempts to incorporate portfolio balance effects into modern DSGEs.

[8]Caballero and Farhi (2014) study QE in a setting with risk neutral and infinitely risk aversion investors and where the central bank's liability plays a special role as a safe asset. Gennaioli et al. (2012) adopts a similar setting to study financial innovation.

2 The Model

I consider a continuous-time stochastic growth model with two goods, consumption and capital goods. The economy is populated by final good producers (firms), two types of market participants (savers and financial intermediaries), and a government (central bank). Final good producers use capital to produce output subject to investment adjustment costs and finance their operations by issuing state contingent liabilities. Investment technology is subject to aggregate shocks. Financial intermediaries (or simply "bankers") and savers trade in dynamically complete markets for aggregate risk.[9] Savers are distinct from intermediaries in two aspects. First, savers are more risk averse than bankers. Second, savers are subject to *market participation shocks* which limit their ability to rebalance their portfolio, as their exposure to risk stays at a predefined level for a (stochastic) period of time. The central bank invests in risky assets financed by its own net worth and riskless reserves. The central bank rebates the proceeds from investment to non-participants. Central bank risk exposure and rebates are defined by policy rules.

In the remainder of this section, I discuss the decision problem of each agent in detail and define the competitive equilibrium. Aggregate conditions are summarized by the vector of aggregate state variables X_t to be explicitly defined later on.

2.1 Final good producers

Firms use capital to produce final goods according to the linear technology

$$Y_t = AK_t \tag{1}$$

Capital evolves according to the law of motion

$$\frac{dK_t}{K_t} = g_t dt + \sigma dZ_t \tag{2}$$

Capital grows at the expected growth rate g_t. In order to achieve a given growth rate g_t the firm must invest $\iota(g_t)K_t$, where $\iota'(\cdot) > 0, \iota''(\cdot) > 0$. This captures the presence of adjustment costs for capital. Importantly, capital accumulation is subject to *investment shocks* with volatility σ. [10]

Firms pay output net of investment as dividends to its shareholders:

$$D_t \equiv AK_t - \iota(g_t)K_t \tag{3}$$

Firms will choose investment to maximize the expected discounted value of dividends. Firms

[9] By an abuse of terminology, I use the expression to mean that dynamic trading spans aggregate risks. I later introduce idiosyncratic mortality shocks that are not spanned by financial markets.

[10] Investment shocks have been identified by the DSGE literature as a main driver of business cycles fluctuations. See Justiniano et al. (2010).

discount future dividends using the *state price density* π_t. The evolution of π_t can be written as

$$\frac{d\pi_t}{\pi_t} = -r_t dt - \eta_t dZ_t \tag{4}$$

where r_t is the instantaneous interest rate and η_t is the market price of risk.

The state price density, interest rate and the market price of risk are all functions of the aggregate state variable X_t: $\pi_t = \pi(X_t)$, $r_t = r(X_t)$, and $\eta_t = \eta(X_t)$. These functions will be determined in equilibrium.

The problem of the firm can then be written as

$$S_t = \max_{g} \mathbb{E}_t \left[\int_t^\infty \frac{\pi_s}{\pi_t} D_s ds \right] \tag{5}$$

subject to (1), (2), (3), (4) and $K_t > 0$.

A version of Hayashi's (1982) theorem hold in this environment, so the value of the firm can be written as $S_t = q_t K_t$, where $q_t = q(X_t)$ evolves according to

$$\frac{dq_t}{q_t} = \mu_{q,t} dt + \sigma_{q,t} dZ_t$$

where $\mu_{q,t}$ and $\sigma_{q,t}$ will be determined in equilibrium.

The return of holding claims in the firm is given by the dividend yield, $\frac{D_t}{S_t}$, plus capital gains, $\frac{dS_t}{S_t}$:[11]

$$dR_t = \underbrace{\left[\frac{A - \iota(g_t)}{q_t} + \mu_{q,t} + g_t + \sigma\sigma_{q,t} \right]}_{\equiv \mu_{R,t}} dt + \underbrace{(\sigma + \sigma_{q,t})}_{\equiv \sigma_{R,t}} dZ_t$$

Notice that the volatility of returns has an exogenous component (σ), due to the volatility of investment shocks, and an endogenous component $(\sigma_{q,t})$, due to endogenous variations in q_t.

2.2 Financial intermediaries

Financial intermediaries solve a standard portfolio problem. They start with net worht $n_{b,0}$ ("b" stands for "bankers") and trade risky claims on firms and a riskless asset. For simplicity, I assume intermediaries do not receive government transfers. Intermediaries choose a process for consumption $(c_{b,t})$ and the share of net worth invested in the risky asset $(\alpha_{b,t})$. It will be the case that in equilibrium, $\alpha_{b,t} > 1$, so intermediaries will issue safe assets to savers and use the proceedings to buy claims on firms.

[11] In order to compute the expected capital gain $\mathbb{E}_t\left[\frac{d(q_t K_t)}{q_t K_t}\right]$, I applied Ito's product rule: $\frac{d(q_t K_t)}{q_t K_t} = \frac{dq_t}{q_t} + \frac{dK_t}{K_t} + \frac{dq_t}{q_t}\frac{dK_t}{K_t}$.

The net worth of intermediaries evolves according to

$$\frac{dn_{b,t}}{n_{b,t}} = [r_t + \alpha_{b,t}(\mu_{R,t} - r_t) - \hat{c}_{b,t}]\, dt + \alpha_{b,t}\sigma_{R,t} dZ_t$$

where $\hat{c}_{b,t} \equiv \frac{c_{b,t}}{n_{b,t}}$.

I will consider a change of variable that will prove useful when characterizing the equilibrium. First, notice that no-arbitrage imply that the excess return on the risky asset is given by

$$\mathbb{E}_t[dR_t] - r_t dt = -\mathbf{Cov}_t\left(\frac{d\pi_t}{\pi_t}, dR_t\right) \iff \mu_{R,t} - r_t = \eta_t \sigma_{R,t} \tag{6}$$

where η_t is (minus) the diffusion term in $\frac{d\pi_t}{\pi_t}$ (see equation (4)).

Define the *risk exposure* of intermediaries as $\sigma_{b,t} \equiv \alpha_{b,t}\sigma_{R,t}$. The decision problem of an intermediary at time t_0 can be written directly in terms of $\sigma_{b,t}$ instead of $\alpha_{b,t}$:

$$V_{b,t_0} = \max_{(c_b, \sigma_b)} U_{b,t_0}(c_b) \tag{7}$$

subject to

$$\frac{dn_{b,t}}{n_{b,t}} = [r_t + \sigma_{b,t}\eta_t - \hat{c}_{b,t}]\, dt + \sigma_{b,t} dZ_t; \qquad n_{b,t} \geq 0 \tag{8}$$

given $n_{b,t_0} > 0$, and equation (6) was used to eliminate $\mu_{R,t} - r_t$.

The formulation highlights the investors care about its risk exposure, regardless if it invests a high share on a low volatility asset or a low share in a high volatility asset, and the excess return it gets per unit of risk. This justifies the term *market price of risk* to denote η_t, which is given to the (instantaneous) Sharpe ratio on the risky investment.[12]

2.3 Households

Households solve a portfolio problem subject to market participation shocks. Households in the *active* state continuously rebalance their portfolio, while households in the *passive* state keep the share invested in the risky asset at a predefined level $\overline{\alpha}$. Active households switch to the passive state with Poisson intensity λ_a and passive investors switch to the active state with Poisson intensity λ_p. Households receive government transfers T_t, choose consumption $c_{j,t}$, and the share invested in the risky asset $\alpha_{j,t}$, given their net worth $n_{j,t}$, for $j \in \{a, p\}$. Passive households are subject to the constraint $\alpha_{p,t} = \overline{\alpha}$.

[12] Another advantage of this formulation is that it encompasses different market structures. For instance, instead of assuming investors trade in a riskless asset and firm's equity, firms could be entirely financed by intermediaries and savers would hold riskless "deposits" and a risky asset issued by intermediaries. The important aspect is that investors can freely trade aggregate risk.

The net worth of households evolves according to

$$\frac{dn_{j,t}}{n_{j,t}} = \left[r_t + \alpha_{j,t}(\mu_{R,t} - r_t) + T_t - \hat{c}_{j,t}\right] dt + \alpha_{j,t}\sigma_{R,t} dZ_t$$

where $\hat{c}_{j,t} \equiv \frac{c_{j,t}}{n_{j,t}}$.

In order to guarantee the existence of a non-degenerate stationary distribution of wealth, investors are subject to mortality risk. Each investor faces a constant hazard rate of death $\kappa > 0$. A mass κ of agents is born every period, so total population is kept constant. Newborn investors are of type b or type s with equal probability, and inherit the net worth of their "parents". Mortality risk will imply the (effective) discount rate is given by $\rho \equiv \hat{\rho} + \kappa$, where $\hat{\rho}$ captures impatience and κ the effect of mortality risk.[13]

Investors have the analogous in continuous-time of Epstein-Zin recursive preferences, as defined by Duffie and Epstein (1992):

$$U_{j,t} = \mathbb{E}_t \left[\int_t^{\infty} f^j(c_{j,s}, U_{j,s}) ds \right] \tag{9}$$

where

$$f^j(c, U) = \rho \frac{(1-\gamma_j)U}{1-\psi^{-1}} \left\{ \left(\frac{c}{((1-\gamma_j)U)^{\frac{1}{1-\gamma_j}}} \right)^{1-\psi^{-1}} - 1 \right\} \tag{10}$$

The coefficient of relative risk aversion is given by γ_j and the elasticity of intertemporal substitution (EIS) is ψ. Intermediaries are assumed to be relatively less risk averse than savers: $\gamma_b \leq \gamma_s$. Notice that while the coefficient of risk aversion depends on the type $j \in \{b, s\}$, the EIS is the same for both groups. Epstein-Zin preferences allow us to focus on heterogeneity of risk aversions while abstracting from differences in the EIS.[14]

2.4 Central Bank

Central bank starts with net worth $n_{cb,0} \geq 0$ and it chooses exposure to aggregate risk $\sigma_{cb,t}$. The central bank rebates the proceeds from its investment to households (T_t). Central bank's balance sheet evolves according to[15]

$$dn_{cb,t} = n_{cb,t}\left[r_t + \sigma_{cb,t}\eta_t - \hat{T}_t\right] dt + n_{cb,t}\sigma_{cb,t} dZ_t \tag{11}$$

[13] See Gârleanu and Panageas (2015) for a derivation of the Epstein-Zin preferences under mortality risk.

[14] As discussed in section 4, Epstein-Zin preferences are also important to obtain the right co-movement between asset prices and risk premium.

[15] To isolate the role of the central bank in reallocating risk in the economy, I assume intermediaries' liability and the central bank's liability (reserves) are perfect substitutes. Hence, the central bank pay interest on reserves r_t in equilibrium. See Drechsler et al. (2014) for a model where reserves play a special role as safe assets.

where $n_{cb,t}\hat{T}_t = T_t$.

Central bank is subject to a No-Ponzi condition:

$$\lim_{T\to\infty} \mathbb{E}_t \left[\frac{\pi_T}{\pi_t} n_{cb,T}\right] \geq 0 \tag{12}$$

I will focus on the case where the central bank chooses policy rules for risk exposure and transfers conditional on the aggregate state variable X_t, i.e., $\sigma_{cb,t} = \sigma_{cb}(X_t)$ and $T_t = T(X_t)$.[16]

2.5 Equilibrium

Definition 1. *An equilibrium is a set of stochastic processes for the interest rate $r = \{r_t : t \geq 0\}$, market price of risk $\eta = \{\eta_t : t \geq 0\}$, state price density $\pi = \{\pi_t : t \geq 0\}$, and the value of the firm $S_t = \{S_t : t \geq 0\}$; aggregate output $Y = \{Y_t : t \geq 0\}$, capital $K = \{K_t : t \geq 0\}$ and investment rate $g = \{g_t : t \geq 0\}$; consumption and risk exposure of intermediaries and active and passive households $c_j = \{c_{j,t} : t \geq 0\}$, $\sigma_j = \{\sigma_{j,t} : t \geq 0\}$, $j \in \{b, a, p\}$; transfers and risk exposure for the central bank $T = \{T_t : t \geq 0\}$, $\sigma_{cb} = \{\sigma_{cb,t} : t \geq 0\}$ such that*

i) g_t solves problem (5) and the value of the objective is S_t.

ii) (c_j, σ_j) solves (7), given (r, η).

iii) (σ_{cb}, T) satisfies (11) given (r, η).

iv) Markets clear:

$$\sum_{j\in\{b,a,p\}} \mu_j c_{j,t} = Y_t - \iota(g_t)K_t$$

$$\sum_{j\in\{b,a,p\}} \mu_j n_{j,t} = S_t - n_{j,t}$$

$$\sum_{j\in\{b,a,p\}} \mu_j n_{j,t}\alpha_{j,t} = S_t - n_{cb,t}\alpha_{cb,t}$$

3 Equilibrium characterization

In this section, I characterize the equilibrium. First, I discuss the role of the two key assumption, heterogeneity in risk-tolerance and in market access.

[16] The central bank effectively acts as an intermediary for households by choosing its risk exposure and rebating the proceeds from the investment. In Silva (2015), I argue that it is not optimal for the central bank to simply replicate the full participation equilibrium in the presence of frictions. When the financial sector is subject to a moral hazard problem, the central bank would deviate from the full participation equilibrium to correct a pecuniary externality. A similar logic applies to environments with aggregate demand externalities.

3.1 The two dimensions of heterogeneity

Before describing the equilibrium conditions in detail, let's consider the role of the two dimensions of heterogeneity: risk tolerance and market participation.

Homogeneous risk tolerances. The following proposition shows that in the absence of heterogeneity in risk tolerance there are no fluctuations in returns or macroeconomic variables (up to scale):

Proposition 1. *Consider an economy with no central bank intervention, i.e.,* $\sigma_{cb,t} = T_t = n_{cb,t} = 0$ *for all* $t \geq 0$. *Suppose* $n_{b,0} = n_{s,0}$. *If* $\gamma_b = \gamma_s \equiv \gamma$, *then*

i. Market price of risk and risk exposures are given by:

$$\eta_t = \gamma\sigma; \qquad \sigma_{b,t} = \sigma_{s,t} = \sigma$$

ii. Growth rate and the price of capital satisfy the conditions:

$$\frac{A - \iota(g_t)}{p_t} = \rho - (1 - \psi^{-1}) \left(g_t - \frac{\gamma\sigma^2}{2} \right); \qquad \iota'(g_t) = p_t$$

iii. Interest rate and consumption-wealth ratios are given by:

$$r_t = \rho + \psi^{-1} g_t - (1 + \psi^{-1}) \frac{\gamma\sigma^2}{2}; \qquad \hat{c}_{b,t} = \hat{c}_{s,t} = \rho - (1 - \psi^{-1}) \left(g_t - \frac{\gamma\sigma^2}{2} \right)$$

The appendix contains closed-form expressions for the price-dividend ratio and the growth rate and provides parameter restrictions for the existence of equilibrium for the case with quadratic adjustment costs.

In the absence of differences in risk tolerance, the economy is essentially deterministic. Of course, aggregate capital is still subject to shocks, so output, consumption, and investment all move in proportion to the capital stock, but scaled variables do not respond to shocks. The assumption that we start at the steady state level of wealth, $n_{b,0} = n_{s,0}$, imply that scaled variables are constant. If we start at a different initial condition, there would be *deterministic* dynamics as the economy converges to the balanced growth path, but still scaled variables would not respond to shocks. The proposition also assumes the central bank does not intervene in the economy. An active central bank is able to affect returns and the macroeconomy provided there is variation in market participation.[17]

The result that risk exposures are the same for both types is more general than stated here. For instance, it does *not* rely on the existence of a representative agent. Scaled variables would still not respond to aggregate shocks if investors had different EIS or if they had access to different idiosyncratic investment opportunities.[18] The key to this result is a combination of the ability of agents to trade aggregate risk with the assumption of equal risk aversion. In this case, both agents will choose the same

[17] However, even in this case there is no balance sheet recession, in the sense the relative net worth of intermediaries and savers will not respond to aggregate shocks.

[18] Gârleanu and Panageas (2015) found a version of this result for the case with different EIS, but no physical investment or differences in investment opportunities. Di Tella (2012) provides a similar result for the case without differences in EIS.

exposure to aggregate risk. Hence, aggregate shocks will affect the balance sheet of both investors equally, so their relative wealth does not change and the same is true for other scaled variables.

Balance sheet recessions, a fall in the growth rate of the economy due to a weakened balance sheet of intermediaries, cannot arise under these assumptions. Brunnermeier and Sannikov (2014) and He and Krishnamurthy (2012) are able to generate balance sheet recessions by restricting aggregate risk sharing. Di Tella (2012) allows for trade in aggregate risk, but balance sheet recessions only arise in the presence of uncertainty shocks.

Proposition 2 shows that risk will be concentrated on the hands of intermediaries, $\sigma_{b,t} > \sigma_{s,t}$, when intermediaries are more risk tolerant. The next section will show how the fact that risk is concentrated on intermediaries can generate balance sheet recessions, even maintaining the assumption of Brownian investment shocks and perfect aggregate risk sharing.

Proposition 2 (Risk concentration). *If $\gamma_s = \gamma_b + \epsilon$, for $\epsilon > 0$ small, then*

$$\sigma_{b,t} - \sigma_{s,t} > 0 \tag{13}$$

Full participation benchmark. The next proposition describes the effect of the balance sheet of the central bank in an economy populated only by intermediaries and savers, and the central bank rebates the profits from investment to savers (denoted by $T_{s,t}$). I discuss the case with transfers to both investors in the appendix.

Proposition 3 (Neutrality result). *Suppose $T_h = 0$, and fix an initial policy rule (σ_{cb}, T_s) and corresponding equilibrium allocation. Consider an alternative policy rule (σ^*_{cb}, T^*_s) that satisfy the central bank's budget constraint.*

i. Prices, consumption, and investment do not change with the central bank's portfolio:

$$(r^*, \eta^*, S^*, c^*_b, c^*_s, g^*) = (r, \eta, S, c_b, c_s, g) \tag{14}$$

ii. Savers exactly offset the portfolio position of the central bank:

$$\sigma^*_{s,t} n^*_{j,t} - \sigma_{s,t} n_{j,t} = -\left(\sigma^*_{cb,t} n^*_{cb,t} - \sigma_{cb,t} n_{cb,t}\right) \tag{15}$$

This result is reminiscent of the neutrality result for open market operations derived by Wallace (1981) .[19] To gain intuition for this result, notice the relevant notion of wealth to savers is given by the sum of financial wealth and the present value of transfers, which is given by the net worth of the

[19] Wallace's result is derived in a monetary overlapping generation economy. Eggertsson and Woodford (2003) shows a similar result in an economy with sticky prices.

central bank in this case. Total wealth is given by $\tilde{n}_{s,t} \equiv n_{s,t} + n_{cb,t}$ and it evolves according to

$$\frac{d\tilde{n}_{s,t}}{\tilde{n}_{s,t}} = [r_t + \tilde{\sigma}_{s,t}\eta_t - \tilde{c}_{s,t}]\, dt + \tilde{\sigma}_{s,t} dZ_t \tag{16}$$

where

$$\tilde{\sigma}_{s,t} \equiv \frac{n_{s,t}}{n_{s,t} + n_{cb,t}}\sigma_{s,t} + \frac{n_{cb,t}}{n_{s,t} + n_{cb,t}}\sigma_{cb,t}; \qquad \tilde{c}_{j,t} \equiv \frac{c_{j,t}}{\tilde{n}_{j,t}}$$

Hence, the relevant risk exposure to the savers is $\tilde{\sigma}_{s,t}$ which includes both financial risk and the riskiness coming from transfers. If government transfers become more risky, perhaps because the central bank increases its exposure to risky assets, this will imply that investors must hold less *financial* risk in order to achieve a desired total risk exposure. Since total risk exposure does not change, then asset prices and macroeconomic variables will not change as well. Importantly, the result does *not* rely on the assumption of complete markets. Even in the presence of risks unspanned by financial markets the neutrality result would hold, provided transfers belong to the space of tradeable assets.

Limited asset market participation breaks this result. If a fraction of agents in the economy is unable to trade in financial markets, then the risk exposure of their total wealth will respond to changes in the central bank portfolio. Hence, unconventional monetary policy works by *redistributing risks* from (marginal) investors to non-participants.[20] If the central bank were to rebate to a fraction of its profits (or losses) to all agents, only the fraction going to non-participants would be non-neutral. Given the neutrality result, I focus on the case where the central bank rebates the proceeds from investment entirely to households.

3.2 Solving the model

Let's go back to the case with differences in risk aversion and in market participation. I will now discuss the solution to the problem of firms, investors, and the determination of prices in equilibrium.

3.2.1 Firms

The HJB equation for the final goods producer gives the pricing condition for capital:

$$r_t + (\sigma + \sigma_{q,t})\eta_t = \max_{g_t} \left\{ \frac{A - \iota(g_t)}{q_t} + \mu_{q,t} + g_t + \sigma\sigma_{q,t} \right\} \tag{17}$$

The optimal investment rate satisfies

$$\iota'(g_t) = q_t \tag{18}$$

Investment is an increasing function of q_t: $g_t = (\iota')^{-1}(q_t)$. The marginal cost of increasing g_t is given by $\iota'(g_t)$ and the marginal benefit is given by increase in the value of the firm, summarized by q_t.

[20] The mechanism is similar to the one in Alvarez et al. (2002) with fixed costs to access asset markets. Borrowing constraints, by impeding the investor to adjust his portfolio at the margin, would work in a similar way. I conjecture that several other frictions would break the neutrality result, for instance, intermittent portfolio rebalancing Chien et al. (2012), rational inattention Sims (2003), and bounded rationality Gabaix (2014).

This will represent the key mechanism connecting asset prices to investment and economic growth.

3.2.2 Investors

Given the homotheticity assumption, the value function of households and intermediaries have a power form $V_{j,t} = V_j(n_{j,t}, X_t)$, $j \in \{b, a, p\}$:

$$V_b(n_{b,t}, X_t) = \frac{(\zeta_t n_{b,t})^{1-\gamma_b}}{1-\gamma_b}; \qquad V_s(n_{s,t}, X_t) = \frac{(\xi_t n_{s,t})^{1-\gamma_s}}{1-\gamma_s} \tag{19}$$

where $\zeta_t = \zeta(X_t)$ and $\xi_t = \xi(X_t)$ follow a diffusion process (to be determined in equilibrium):

$$\frac{d\zeta_t}{\zeta_t} = \mu_{\zeta,t}dt + \sigma_{\zeta,t}dZ_t; \qquad \frac{d\xi_t}{\xi_t} = \mu_{\xi,t}dt + \sigma_{\xi,t}dZ_t;$$

I refer to (ζ_t, ξ_t) as *net worth multipliers*. Net worth multipliers are related to the marginal utility of wealth. They capture the fact that the marginal utility of wealth depends on the level of returns. For instance, if returns are expected to be low for a reasonable amount of time in the future, this will hurt those who rely on financial assets to finance future consumption. Hence, an additional unity of wealth in those states will not increase utility by the same amount compared to states where returns are high.[21] This will be reflected in a lower value for the net worth multiplier.

After some rearrangement, the HJB equation for intermediaries can be written as:

$$0 = \max_{\hat{c}_b, \sigma_b} \frac{\rho}{1-\psi^{-1}} \left[\left(\frac{\hat{c}_{b,t}}{\zeta_t} \right)^{1-\psi^{-1}} - 1 \right] + r_t + \sigma_{b,t}\eta_t - \hat{c}_{b,t} + \mu_{\zeta,t} - \frac{\gamma_b}{2} \left(\sigma_{b,t}^2 - 2\frac{1-\gamma_b}{\gamma_b}\sigma_{\zeta,t}\sigma_{b,t} + \sigma_{\zeta,t}^2 \right) \tag{20}$$

and an analogous condition holds for savers.

The optimal risk exposure is given by

$$\sigma_{b,t} = \underbrace{\frac{\eta_t}{\gamma_b}}_{\text{myopic}} + \underbrace{\frac{1-\gamma_b}{\gamma_b}\sigma_{\zeta,t}}_{\text{hedging}}; \tag{21}$$

The optimal portfolio decision has two components: a *myopic demand* and a *hedging demand*. The myopic demand coincides with the portfolio of a one-period mean-variance investor (hence the name "myopic"). It equals the market price of risk (or Sharpe ratio) times the risk tolerance. The fact that intermediaries are more sensitive to changes in the market price of risk than savers will be important when considering the effects of unconventional monetary policy.

Hedging demand captures deviations from the mean-variance portfolio due to variations in investment opportunities. If returns were constant, then $\sigma_{\zeta,t} = 0$ and intermediaries would act as mean-

[21] It is usually argued that zero interest rates combined with quantitative easing hurt savers and those planning future retirement (see Jeff Cox, "FED policies have cost savers $470 billion: Report", CNBC, March 26, 2015).

variance investors. In equilibrium, returns will actually be *countercyclical*, so $\sigma_{\zeta,t} < 0$.[22] This captures the fact that after a negative shock asset prices fall, so the marginal utility of wealth increases for intermediaries. If we assume $\gamma_b < 1$, intermediaries react to countercyclical returns by reducing risk-taking. Since returns are high after a negative shock, intermediaries shift resources to those states, taking less risk ex-ante.

If $\gamma_b > 1$, then intermediaries would take more risk than a mean-variance investor. The reason is that instead of trying to send resources to states of nature where returns are high, intermediaries would do the opposite. Returns are already high after a negative shock, so less resources are needed to achieve the same level of utility. This *income effect* will dominate when $\gamma_b > 1$.[23]

Consider now the optimal consumption-wealth ratio:

$$\hat{c}_{b,t} = \rho^{\psi}\zeta_t^{1-\psi}; \qquad \hat{c}_{s,t} = \rho^{\psi}\xi_t^{1-\psi}; \tag{22}$$

If EIS is equal to one, $\psi = 1$, then the consumption-wealth ratio is constant and equal to ρ. When $\psi \neq 1$, consumption-wealth ratio will respond to changes in investment opportunities. Plugging (22) into (20), we obtain

$$\rho^{\psi}\zeta_t^{1-\psi} = \psi\rho + (1-\psi)\left[\underbrace{r_t + \sigma_{b,t}\eta_t}_{\text{return on portfolio}} + \underbrace{\mu_{\zeta,t}}_{\text{change in inv. opp.}} - \underbrace{\Phi_{b,t}}_{\text{precautionary sav.}}\right] \tag{23}$$

where

$$\Phi_{b,t} \equiv \frac{\gamma_b}{2}\left[\sigma_{b,t}^2 + 2\frac{\gamma_b - 1}{\gamma_b}\sigma_{\zeta,t}\sigma_{b,t} + \sigma_{\zeta,t}^2\right]$$

As in the traditional Fisherian analysis, the effect of interest rates on consumption generates income and substitution effects and the EIS determines which effect dominates. The expression above shows that a similar logic apply not only to the riskless return, but to the expected return on the portfolio adjusting for risk and expectations of changes in future returns.[24]

3.2.3 Market price of risk and interest rate

Define the share of *private wealth* held by intermediaries, x_t, and the share of *total wealth* held by the central bank, w_t:

$$x_t = \frac{n_{b,t}}{n_{b,t} + n_{s,t}}; \qquad w_t = \frac{n_{cb,t}}{n_{b,t} + n_{s,t} + n_{cb,t}}$$

[22]See discussion in section 2 for a discussion of the countercyclicality of returns.

[23]The available empirical evidence on the risk-management of banks, in particular on the use of interest rate derivatives, seems to indicate the substitution effect dominates (see Begenau et al. (2013)).

[24]For instance, suppose returns are expected to improve and $\mu_{\zeta,t} > 0$. On the one hand, the investor may save more to have resources in the future when returns are high (substitution effect). On the other hand, it may save less since less resources are necessary to achieve the same level of utility in the future (income effect). If $\psi > 1$ the substitution effect dominates.

Using these definitions, we can write the market clearing condition for risk exposures as

$$x_t \sigma_{b,t} + (1 - x_t)\sigma_{s,t} = \omega_t^r (\sigma + \sigma_{q,t}) \tag{24}$$

where

$$\omega_t^r \equiv \frac{1 - \frac{n_{cb,t}\sigma_{cb,t}}{\sigma_{S,t} S_t}}{1 - w_t}$$

The term ω_t^r measures the *net asset supply* to market participants. It equals the share of aggregate risk (per unit of wealth) held by private agents. In the absence of a central bank ($n_{cb,t} = 0$) or when the central bank's share of aggregate risk equals its share of wealth ($\sigma_{cb,t} = \sigma_{S,t}$), net asset supply equals one. If the central bank decides to hold proportionally more risk than its wealth share, then market participants will hold relatively less risk ($\omega_t^r < 1$). Hence, an expansion of the balance sheet where the central bank buys risky assets by issuing reserves will reduce the net asset supply to sophisticated investors.

Plugging in risk exposures from (21) into (24), we obtain an expression for the market price of risk:

$$\eta_t = \underbrace{\gamma_t}_{\text{agg. risk aversion}} \left[\underbrace{\omega_t^r (\sigma + \sigma_{q,t})}_{\text{net supply of risk}} - \underbrace{\left(x_t \frac{1 - \gamma_b}{\gamma_b} \sigma_{\zeta,t} + (1 - x_t) \frac{1 - \gamma_s}{\gamma_s} \sigma_{\xi,t} \right)}_{\text{avg. hedging demand}} \right] \tag{25}$$

where γ_t is the aggregate risk aversion

$$\gamma_t = \left(\frac{x_t}{\gamma_b} + \frac{1 - x_t}{\gamma_s} \right)^{-1} \tag{26}$$

The market price of risk is the product of aggregate risk aversion with the difference between the net supply of risk and the average hedging demand. Hence, periods where intermediaries are relatively less capitalized, so aggregate risk aversion is high, will tend to have a high market price of risk. Similarly, this suggests that reductions in the supply of risk associated with the expansion of the balance sheet of the central bank will reduce the price of risk. Finally, if average hedging demand is high, then a smaller market price of risk will be required to induce investors to hold the net supply of risk.

Consider now the market clearing condition for goods:

$$x_t \hat{c}_{b,t} + (1 - x_t)\hat{c}_{s,t} = \omega_t^d \frac{A - \iota(g_t)}{q_t} \tag{27}$$

where

$$\omega_t^d \equiv \frac{1 - \frac{T_t}{D_t}}{1 - w_t}$$

The term ω_t^d captures the impact of central bank policy on the goods market. In the absence of a central bank or if the central bank rebates to workers all dividends received, then $\omega_t^d = 1$. If the central bank decides to reduce the size of its balance sheet by transferring relatively more resources to workers, this reduce the supply of goods to sophisticated investors ($\omega_t^d < 1$).

Plugging in the expression for the consumption-wealth ratio into (27), we obtain an expression for the interest rate:

$$\psi\rho + (1-\psi)\left[r_t + \omega_t^r(\sigma + \sigma_{q,t})\eta_t + \mu_t - \Phi_t\right] = \omega_t^d \frac{A - \iota(g_t)}{q_t} \tag{28}$$

where

$$\mu_t \equiv x_t\mu_{\zeta,t} + (1-x_t)\mu_{\xi,t}; \qquad \Phi_t \equiv x_t\Phi_{b,t} + (1-x_t)\Phi_{s,t};$$

The equation for the interest rate can be interpreted as representing the aggregate demand (in the left-hand side) and the aggregate supply for goods (in the right-hand side), both normalized by total wealth. The effect of interest rate on aggregate demand depends crucially on the EIS: if $\psi > 1$, then an increase in the interest rate decreases aggregate demand (the opposite is true when $\psi < 1$).

3.3 Markov Equilibrium

I will focus on a Markov equilibrium on the state variable $X = (x, w) \in [0,1]^2$, so equilibrium prices $(r(X), \eta(X), q(X))$ and net worth multipliers $(\xi(X), \zeta(X))$ are all functions of X_t. Moreover, in a Markov equilibrium the central bank chooses policy rules that depends only on X_t. Instead of specifying the central bank policy in terms of $(\sigma_{cb,t}, T_t)$, I will assume without loss of generality the central chooses directly (ω_t^r, ω_t^d):

$$\omega_t^r = \omega^r(X_t); \qquad \omega_t^d = \omega^d(X_t) \tag{29}$$

I impose two constraints on policy rules: i) $\omega^d(x, w) > 0$ for all $(x, w) \in [0,1]^2$; ii) $\omega^r(x, 0) = 1$ for all $x \in [0,1]$. The first condition guarantees that consumption of sophisticated investors is always positive. The second constraint says the central bank cannot have infinite leverage, i.e., if the net worth of the central bank goes to zero, then asset holdings must also go to zero.

The next proposition characterizes the law of motion of X:

Proposition 4. *The state variables (x, w) evolve according to*

$$dx_t = \mu_{x,t}dt + \sigma_{x,t}dZ_t; \qquad dw_t = \mu_{w,t}dt + \sigma_{w,t}dZ_t;$$

i. The drift of x and w is given by

$$\mu_{x,t} = x_t(1-x_t)\left[(\sigma_{b,t} - \sigma_{s,t})\left(\eta_t - \omega_t^r(\sigma + \sigma_{q,t})\right) + \hat{c}_{s,t} - \hat{c}_{b,t}\right] - \kappa\left(x_t - 0.5\right)$$
$$\mu_{w,t} = (1-w_t)\left[(1-\omega_t^r)\left(\sigma + \sigma_{q,t}\right)\left(\eta_t - (\sigma + \sigma_{p,t})\right) - (1-\omega_t^d)\frac{A - \iota(g_t)}{q_t}\right]$$

ii. The diffusion of x and w is given by

$$\sigma_{x,t} = x_t(1-x_t)(\sigma_{b,t} - \sigma_{s,t}); \qquad \sigma_{w,t} = (1-w_t)(1-\omega^r_t)(\sigma + \sigma_{q,t}); \tag{30}$$

The appendix provides the expression for (σ_x, σ_w) and (μ_x, μ_w) in terms of (ζ, ξ) and its derivatives.

The proposition describes the law of motion of the state variables. The assumption that investors have finite lives guarantees the extremes $x_t = 0$ or $x_t = 1$ are not absorbing states. In particular, $\mu_{x,t} < 0$ for $x_t = 1$ and $\mu_{x,t} > 0$ for $x_t = 0$. The assumption that future generations do not necessarily inherit the type of their "parents" imply that no type will eventually hold all the wealth in the economy.

The proposition shows that the diffusion of x depends on the relative risk exposure of intermediaries and savers $\sigma_{b,t} - \sigma_{s,t}$. Since the volatility of q_t is qiven by $\sigma_{q,t} = \frac{q_{x,t}}{q_t}\sigma_{x,t}$, the amount of endogenous volatility depends on the degree of risk concentration.

A corollary of the proposition above is that if $\omega^r_t = \omega^d_t = 1$, then w_t is constant, and the equilibrium allocation is identical to an economy without the central bank. This suggests the importance of the central bank operating leveraged in order to affect the economy.

Corollary 1. *Suppose $\omega^r(x, w_0) = \omega^d(x, w_0) = 1$ for all $x \in [0, 1]$, then $w_t = w_0$ for all $t \geq 0$. Moreover, equilibrium conditions coincide with the ones in a unregulated equilibrium without the central bank $(w_0 = 0)$.*

3.4 Numerical solution and calibration

The functions $\zeta(x, w)$ and $\xi(x, w)$ can be obtained by solving a system of partial differential equations (PDEs). For this we need two conditions that can be expressed only in terms of the net worth multipliers and its derivatives. Hence, we need to compute the remaining equilibrium conditions as functions of (ζ, ξ). In the appendix B, I discuss the algorithm used to solve the system of PDEs.

I adopt the following calibration. The level of technology A is set to $1/3$ in order to match a capital-output ratio of 3. Depreciation rate is set to $\delta = 0.05$. Investment adjustment costs are assumed to be quadratic $\iota(g) = \phi_0(g+\delta) + \frac{\phi_1}{2}(g+\delta)^2$, where ϕ_0 and ϕ_1 are chosen to match an average investment rate of 20% and average growth rate of 2%.[25] Volatility of aggregate output is set to match the (time-integrated) one-year volatility of output 2.33% in a post-war sample period. I set the EIS to $\psi = 2$, a common value found in the literature.[26] There is little guide for the choice of the risk aversion of intermediaries. I will focus on the case where the substitution effect dominates in the portfolio choice $(\gamma_b < 1)$ and intermediaries have preferences for early resolution of uncertainty $(\gamma_b > \psi^{-1} = 1/2)$. I chose $\gamma_b = 0.7$ but other values on this range generate similar results. The risk aversion of savers is

[25] The search for the parameters is subject to the conditions (A.1) for $\gamma = \gamma_b$ and $\gamma = \gamma_s$. These conditions guarantee the existence of equilibrium in the polar cases where a single type of sophisticated investor holds all private wealth.

[26] It is also a value commonly found in empirical studies that focus on the EIS for market participants. Vissing-Jørgensen and Attanasio (2003) explicitly distinguishes between EIS and risk aversion and find values between 1 and 2. Gruber (2013) estimates an EIS of 2 using tax data. Kapoor and Ravi (2010) estimates an EIS of 2.2 exploring a change in banking regulation in India.

set to $\gamma_s = 30$. This will generate a value for the average risk aversion around 1.4 and 3.6 during 90% of the time at the stationary distribution for x_t in the laissez-faire. Mortality rate is given by $\kappa = 0.02$ and $\tilde{\rho} = 0.001$ such that $\rho \approx 0.02$. For the numerical solution, I extend the model to allow for different sizes of intermediaries and savers and I set the share of intermediaries to $\theta_b = 0.1\%$.

4 Balance Sheet Recessions

In this section, I consider how unconventional monetary policy can be used to counteract the effects of balance sheet recessions. I first show how balance sheet recessions can emerge in a laissez-faire equilibrium. I discuss then the choice of policy rules for the central bank, the impact of central bank policy on asset prices and growth, and how the effectiveness of the policy vary with the state of the economy and the size of the intervention.

4.1 The Laissez-Faire Equilibrium

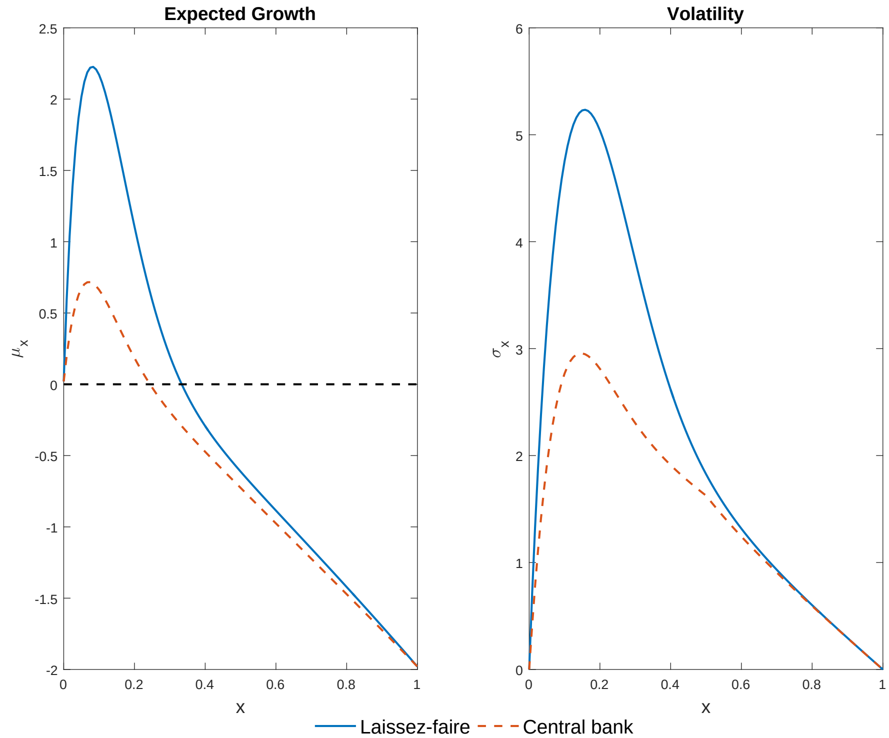
Let's consider initially the case without the central bank, i.e., $\omega_t^r = \omega_t^d = 1$ and $w_0 = 0$. The evolution of the state variable x_t is given in figure 1, where $\mu_{x,t}$ and $\sigma_{x,t}$ are plotted as functions of x_t. The drift of x_t is positive for low levels of x_t and negative for high values of x_t. The point where it crosses zero is the *stochastic steady state*, the point of attraction of the system in the absence of shocks. Importantly, the diffusion term is always positive. Hence, from (30) and consistent with proposition 2, intermediaries are always more exposed to risk than savers.

The fact that intermediaries operate leveraged in equilibrium, so x_t is positively exposed to risk, will imply the economy is subject to *balance sheet recessions*. Figure 2 shows the price of capital q_t, the market price of risk η_t, interest rate r_t, and volatility of returns $\sigma + \sigma_{q,t}$ as functions of the relative strength of intermediaries' balance sheet (x_t).

The key feature driving variation in assets prices is a *countercyclical aggregate risk aversion*. Since intermediaries are more exposed to risk than savers, their share of wealth fall after a negative shock. Average risk aversion in the economy rises (see (26)). This will push the market price of risk up and interest rates down.

Here is the intuition behind figure 2. After a negative aggregate shock, the share of wealth of intermediaries will fall, given their higher exposure to risk. Savers will have to absorb a higher fraction of the risk and, since they are more risk-averse, the market price of risk will have to increase. The interest rate will fall, as the average precautionary motive becomes stronger as savers become relatively more important. The effect on the price is, in principle, ambiguous. The assumption that $\psi > 1$ plays a role to determine which effect dominates. Given a high elasticity of substitution, a small movement on interest rates is enough to restore equilibrium, so the price of capital will fall after a negative shock. If $\psi < 1$, a stronger response of interest rates would be required and the price of capital would actually

Figure 1: Law of Motion of x_t



Note: Solid (dashed) line represents laissez-faire (central bank) equilibrium. Solution with the central bank evaluated at the ergodic mean of w_t. Drift and diffusion of x_t in percentage points.

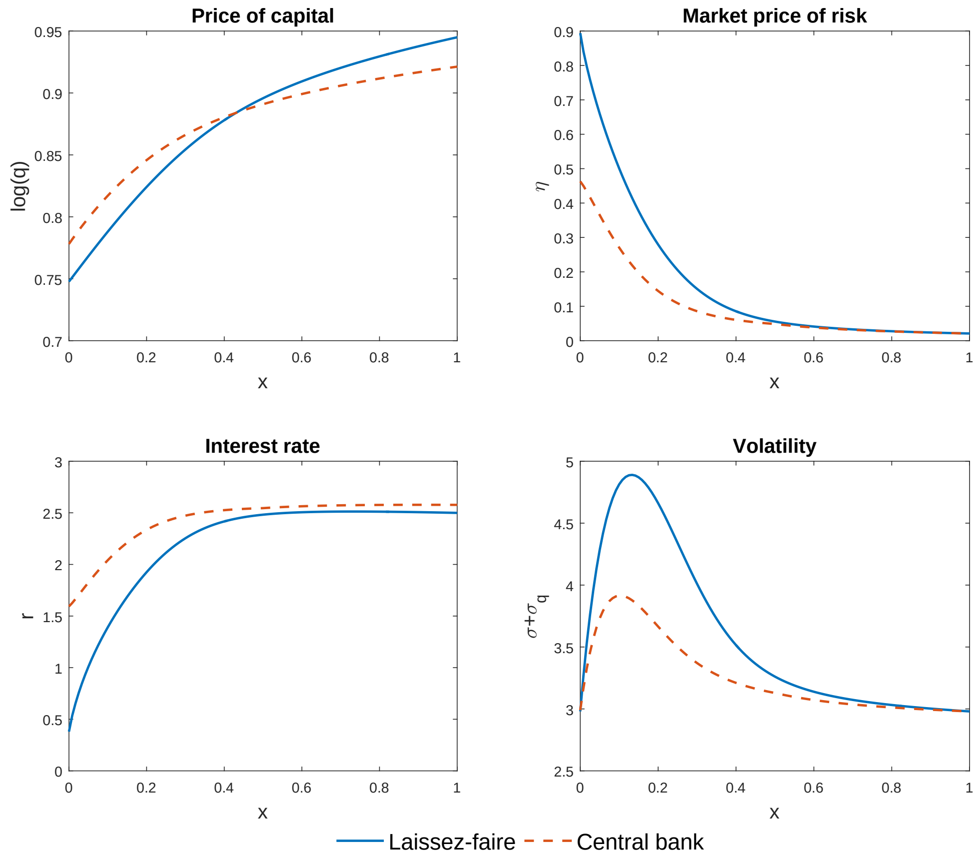
fall after a reduction in x_t.

Volatility is a non-monotonic function of x_t. It inherits this pattern from the diffusion of the state variable. The reason is that if the net worth of an agent is sufficiently close to zero, then her wealth will not respond much to shocks (in the limit as the net worth is zero, there is no response to shocks). Even though the response is non-monotonic, for values of x_t around the stochastic steady state, a balance sheet recession (a reduction in x_t) will be associated with an increase in endogenous volatility.

Balance sheet recessions can be understood as the consequence of a *financial fire sale*. As the risk bearing capacity of intermediaries is reduced after a negative shock, high risk aversion agents will hold more of the risk in equilibrium, requiring a higher risk compensation. Savers make the role of a *second-best owner* of the asset, similar to farmers in Kiyotaki and Moore (1997). Importantly, the fire sale affects asset prices through changes in the *discount rate*. In contrast to the fire sales in Kiyotaki and Moore (1997) and Brunnermeier and Sannikov (2014), where the physical asset changes hands and asset prices are affected due to a direct reduction in the dividend.[27]

[27] See Cochrane (2011) for a review of the literature documenting the importance of variations in discount rates to explain movements in asset prices.

Figure 2: Balance Sheet Recessions



Note: Solid (dashed) line represents laissez-faire (central bank) equilibrium. Solution with the central bank evaluated at the ergodic mean of w_t. Interest rate and volatility in percentage points.

4.2 The Effects of Unconventional Monetary Policy

4.2.1 Policy rules

Consider the economy with a central bank. To capture the idea that unconventional monetary policy is a policy instrument typically used during crises, $\omega^r(\cdot, w)$ is assumed to be increasing in x:

$$\omega^r(x, w) = \beta_0^r(w) + \beta_1^r(w) \min\{x, x^*\} \tag{31}$$

where $\beta_0^r(w) \leq 1$ and $\beta_0^r(w) + \beta_1^r(w)x^* = 1$.

Remember a low of value of $\omega^r(\cdot)$ means the central bank is holding proportionally a large fraction of risk.[28] Hence, when intermediaries are relatively less capitalized ($x_t < x^*$) the central bank will intervene and reduce the net asset supply ω^r. When intermediaries are relatively well capitalized, the central bank will keep the net asset supply at the laissez-faire value $\omega^r = 1$.

Since the central bank does not intervene if it has no net worth, the coefficients in the policy rule

[28] More precisely, $\omega^r(\cdot) < 1$ if and only if the fraction of risk held by the central bank exceeds w_t.

satisfy the condition $\beta_0^r(0) = 1$ and $\beta_1^r(0) = 0$. In the calibrated example, the coefficients in the policy rule satisfy $\beta_0^r(w) = 0.5$ and $\beta^r(w) = 1$ for $w \geq w^*$, where $w^* = 0.01$. For $w < w^*$, the coefficients are linearly interpolated: $\beta_0^r(w) = 1 - 0.5\frac{w}{w^*}$ and $\beta_1^r(w) = \frac{w}{w^*}$. The precise value of w^* and specification of the coefficients for $w < w^*$ have only a minor impact in the solution provided w^* is sufficiently small.

The proposed policy is meant to illustrate the effects of an aggressive policy, in particular, in very low probability states with low values of x_t. In a stationary distribution of the laissez-faire equilibrium, the first quartile of x_t is 0.3 and the third quartile is 0.42. Hence, the system spends most of the time around moderate values of x_t. Even extreme events do not reach values of x around 0. For instance, the 5th percentile is equal to 0.18. However, promises to intervene in these extreme events will have an impact on prices.

The policy rule $\omega^d(x, w)$ is assumed to be linear in w and independent of x:

$$\omega^d(x, w) = \beta_0^d + \beta_1^d w \tag{32}$$

where $\beta_0^d = 1.01$ and $\beta_1^d = -0.02$ in the calibrated example.

The role of the variation in ω^d is to guarantee that wealth of the central bank will return to the interior of the state space if ever reaches the boundaries of the system.

4.2.2 Unconventional Monetary Policy and Balance Sheet Recessions

Given the specified policy rules, we can compute the equilibrium in the presence of the central bank. Figure 1 shows the impact of unconventional monetary policy in the law of motion of x_t. Both the drift and the diffusion are uniformly reduced by the policy. The effect on the volatility of x_t and its connection with the concentration of risks in the financial sector will be discussed in detail in the next section.

Figure 2 shows how the policy of the central bank affects asset prices. Unconventional monetary policy reduces the market price of risk. As the central bank expands its balance sheet, the net supply of risk to sophisticated investors falls and, from (25), contributes to the reduction in η. Volatility of returns is reduced as the volatility of the state variable goes down. This reduction in volatility will contribute to the reduction in the market price of risk even in periods where $\omega^r = 1$.

The effect of asset purchases will be stronger the weaker the balance sheet position of intermediaries. There are two reasons for this. First, given the assumed policy rule, the central bank will intervene more in bad times. Second, demand for assets is more elastic when intermediaries are relatively well capitalized. Intermediaries tend to respond more strongly to changes in returns than savers. When intermediaries are undercapitalized, savers must bear most of the risk, and aggregate demand for risk will be relatively insensitive to returns. Hence, a higher reduction in returns is required to accommodate a reduction in the supply of risk caused by central bank policy.

The reduction in the return to the risky asset, by the intertemporal substitution channel, and the reduction in volatility, by the precautionary savings channel, will both tend to increase aggregate demand (or equivalently, reduce the incentive to save). In order to restore equilibrium, the riskless interest rate goes up. For low levels of x_t, the reduction in the risk premium will dominate the increase in the riskless rate and the price of capital will go up. However, for high values of x_t the interest rate will dominate and the price of the asset will go down.

Hence, unconventional monetary policy ameliorates the effects of balance sheet recessions in crisis, at the cost of reducing the growth rate in normal times. The central bank is able to reduce risk premium and volatility while it boosts investment and growth during a crisis. However, it reduces economic growth in booms compared to the laissez-faire economy.

5 Financial Stability

This section shows how unconventional monetary policy affect financial stability. First, I discuss how the concentration of risk in the financial sector responds to changes in central bank policy. I compute the stationary distribution for asset prices and show how the model generates endogenous "disasters", i.e, a relatively high probability of negative tail events. I show that unconventional monetary policy reduces tail risk but it reduces average growth rate in the economy.

5.1 Intermediaries' Risk Taking Decision

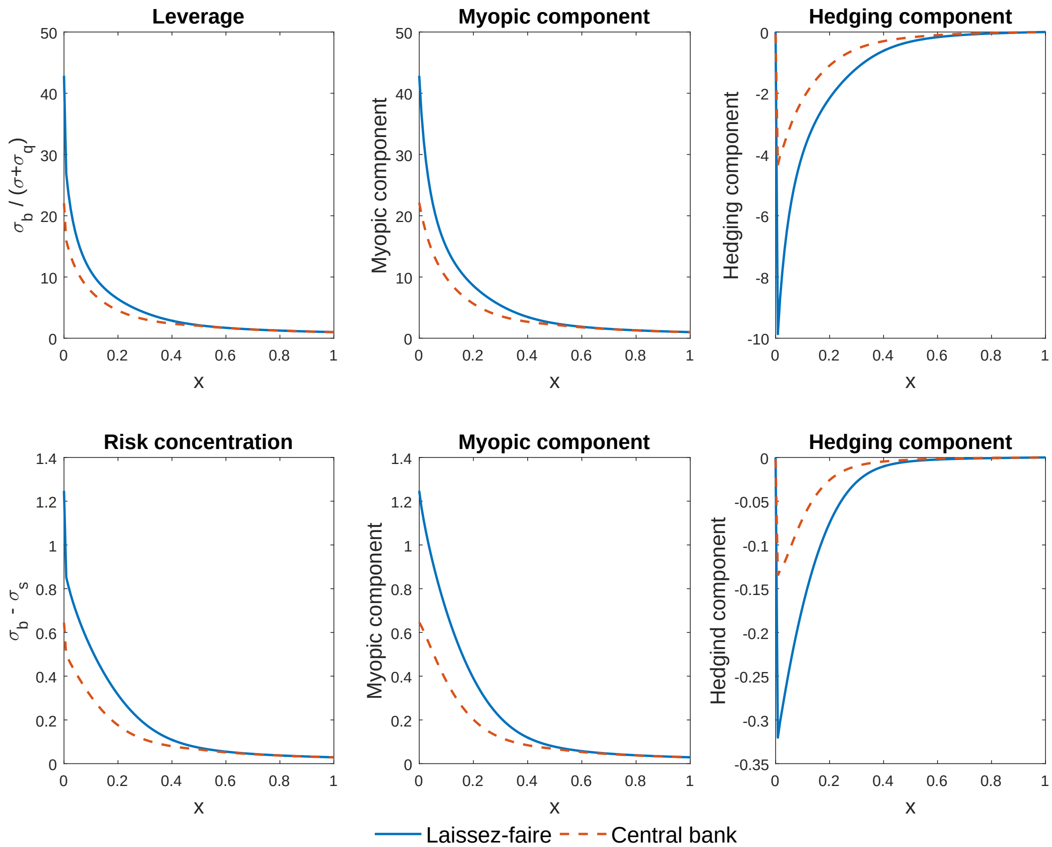
Unconventional monetary policy will affect the leverage decision of intermediaries, as the incentives to hold risky assets will respond to central bank's asset purchase.

Figure 3 shows intermediaries' leverage and risk concentration as a function of x_t for the laissez-faire economy and the economy with the central bank.[29] Notice that, compared to the homogeneous preferences benchmark (where leverage is always equal to one), the heterogeneous agent economy generates significant concentration of risk. The figure also shows the myopic and hedging component, as defined in (21), as the central bank policy will have different effects in the different components.

The hedging component is negative for all $x \in [0, 1]$. The reason is that intermediaries anticipate that after a negative shock returns will increase. Intermediaries hedge against these changes in returns and reduce risk taking ex-ante. Consider now the effect of asset purchases by the central bank. Central bank policy reduces returns relatively more in bad times. The incentives for intermediaries to hedge will then be reduced, causing them to take more risk. Since risk-taking increases after a reduction in

[29] Leverage in this setting is simply risk exposure divided by volatility $\frac{\sigma_{b,t}}{\sigma+\sigma_{q,t}}$. Risk concentration is the difference between the exposure of intermediaries and savers: $\sigma_{b,t} - \sigma_{s,t}$.

Figure 3: Intermediares' Leverage and Risk Concentration



Note: Solid (dashed) line represents laissez-faire (central bank) equilibrium. Solution with the central bank evaluated at the ergodic mean of w_t. Leverage (assets over net worth) equals risk exposure over volatility.

returns, I refer to this effect as the *hedging effect*.[30]

Consider now the myopic component. As the market price of risk is reduced, myopic demand falls with the central bank intervention. Importantly, myopic demand falls *more* to intermediaries than savers. The reason is that intermediaries are relatively more risk tolerant, which imply they are also more sensitive to changes in η_t. Therefore, risk concentration will tend to fall with the central bank policy. I refer to this effect as the *return sensitivity effect*.

Another way of looking at the return sensitivity effect is to consider who will the central bank buy assets from. Since savers have high risk aversion, they don't respond very strongly to changes in returns. As the central bank buys risky assets, a given drop in returns would not be enough to induce savers to change its portfolio by much, but it would be enough to induce intermediaries to sell. Hence, most of the assets will flow from intermediaries to the central bank. This will tend to reduce risk exposure of intermediaries relative to savers.

[30]The leverage decision of savers have a similar pattern, but it is quantitatively smaller. Hence, the *relative* hedging demand behaves similarly to the hedging demand of intermediaries.

Figure 3 shows that the return sensitivity effect dominates, so leverage of intermediaries and risk concentration falls. One intuition for this result is that the hedging effect arises because the volatility falls, so returns will not increase as much after a negative shock. But if the hedging effect were to dominate, risk concentration and volatility would increase, contradicting the fact that volatility must go down to generate the hedging effect.[31] This suggests that the return sensitivity effect should dominate the hedging effect.

5.2 Stationary distributions

So far we considered how unconventional monetary policy can affect the economy if the economy is in a crisis. I will now focus on how the central bank balance sheet can affect the *likelihood* of future crisis.

Figure 4 shows the stationary distribution of the growth rate of capital for the laissez-faire economy and for the economy with the central bank. The left panel shows the probability density function (PDF) for the two economies and the right panel shows the tail behavior of the stationary distribution measured by the probability of being a given number of standard-deviations below the mean.

An important feature of these distributions is that they present *negative skewness* and *excess kurtosis*, as can be seen in table 1. This means the economy is subject to (left) tail risk. [32]

Table 1: Summary Statistics of Stationary Distribution for g_t

Economy	Mean	Std. Dev.	Skewness	Kurtosis
Laissez-faire	1.3%	0.2%	-1.45	5.64
Central bank	1.2%	0.1%	-1.09	3,98

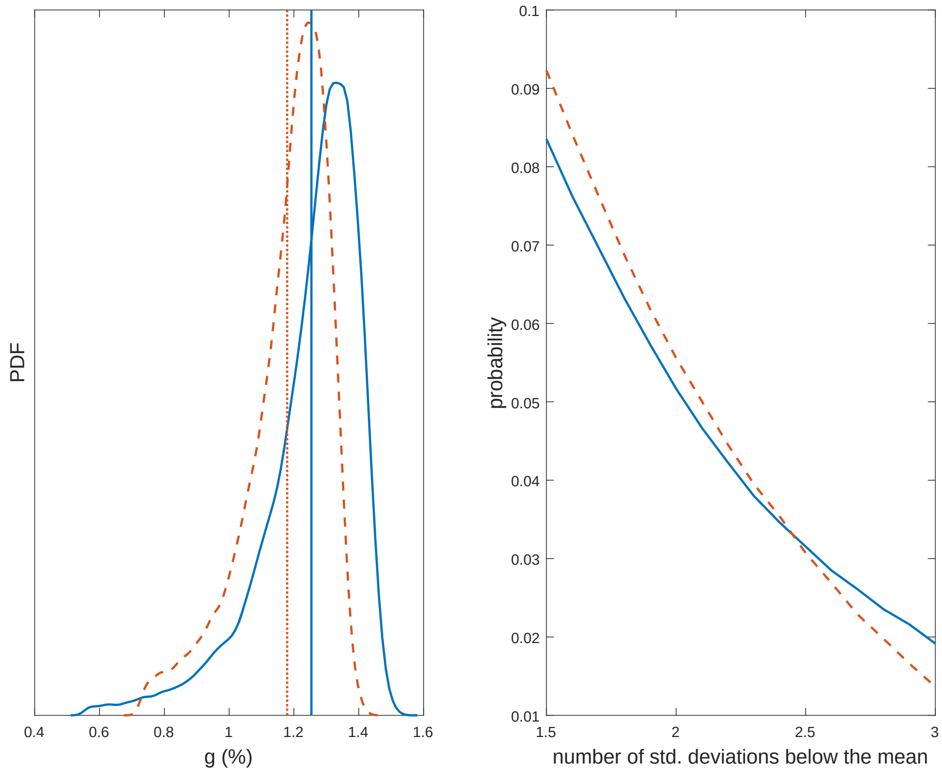
The economic mechanism that generates tail risk is related to the return sensitivity effect described in section 5.1. Suppose that initially intermediaries are relatively well capitalized and the price of risk is low. After a negative shock, wealth is redistributed towards savers and the aggregate demand for risk falls generating an increase in the price of risk. However, when intermediaries are initially well capitalized the effect on the aggregate demand for risk is small, since the difference between the portfolio of intermediaries and savers is also small.[33] Suppose now that intermediaries have low risk bearing capacity (low x_t). The same redistribution of wealth between intermediaries and savers have now a big effect on the aggregate demand for risk. The reason is that when returns are high, intermediaries have a greater incentive to hold risk so a reduction in their risk bearing capacity has a big impact on the demand for risk. In this situation, the same redistribution between intermediaries and savers will have a big impact on the price of risk and the price of capital. This asymmetric response of the price of

[31] This argument is incomplete since it ignores the effect of the volatility of w_t on returns.

[32] In particular, extreme negative events are much more likely than in a normal distribution. The probability of a negative 2 standard-deviation event is more than twice the one for the normal distribution and the difference is even higher for more extreme events.

[33] As shown in figure 3, risk concentration falls with share of wealth of intermediaries x_t.

Figure 4: Stationary Distribution



Note: Solid (dashed) line represents laissez-faire (central bank) equilibrium. Solution with the central bank evaluated at the ergodic mean of w_t.

capital will translate in an asymmetric response of the growth rate of capital through equation (18).

Consider now the economy with a central bank. As described above, risk concentration falls with central bank policy, especially for low levels of x_t. Hence, the system will present less of a asymmetric response to shocks and central bank policy reduces skewness and kurtosis. Hence, if we measure financial stability by left-tail risk, we can conclude that asset purchases by the central bank *enhances* financial stability.

In contrast, the average growth rate in the economy falls. As discussed in section 4, asset purchases have an ambiguous effect on the price of risky assets. In particular, the price of the risky asset falls if intermediaries are relatively well capitalized. As the incentive to save falls with the policy, interest rates will increase, reducing the incentive to invest. Hence, asset purchases by the central bank reduces average growth rate in the economy.

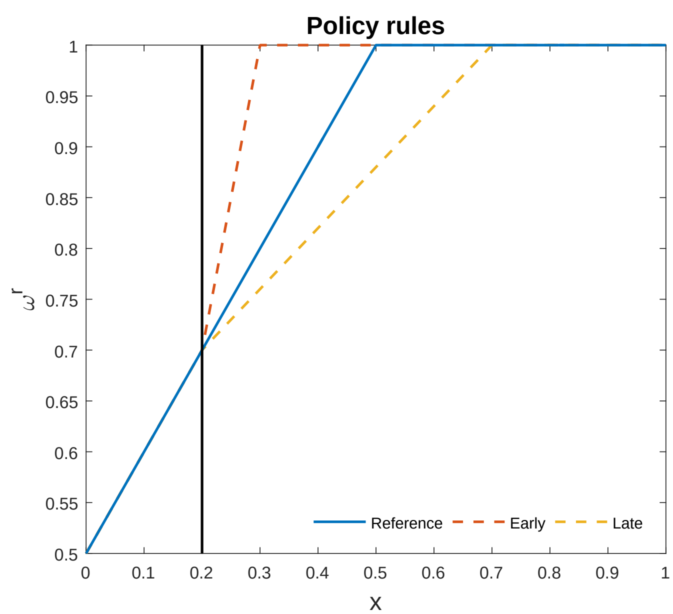
6 Exit Strategies

Consider now the role of different exit strategies. I will focus on state-contingent rules where the central bank unwind its portfolio according to the balance sheet position of intermediaries. In particular, there is *reference strategy*, which correspond to the policy rule we have been analyzing so far, and two alternative rules, an *early exit* and a *late exit* strategy. The first will completely unwind its portfolio when $x = 0.3$ and the second when $x = 0.7$. Figure 5 plots the policy rules.

Figure 6 shows the market price of risk and the price of capital for early and late strategies *relative* to the reference strategy. Over the region $0 \leq x \leq 0.2$ where all policy rules coincide, the price of risk is smaller for the early strategy and higher for the late strategy. Here is the intuition: by promising to sell faster, conditional on the balance sheet of intermediaries, the central bank makes returns in good times relatively more attractive. Hence, intermediaries will have an incentive to take more risk. Given this higher demand for risk, the price of risk must fall. Therefore, risk premium is smaller under the early strategy, for the region policies coincide.

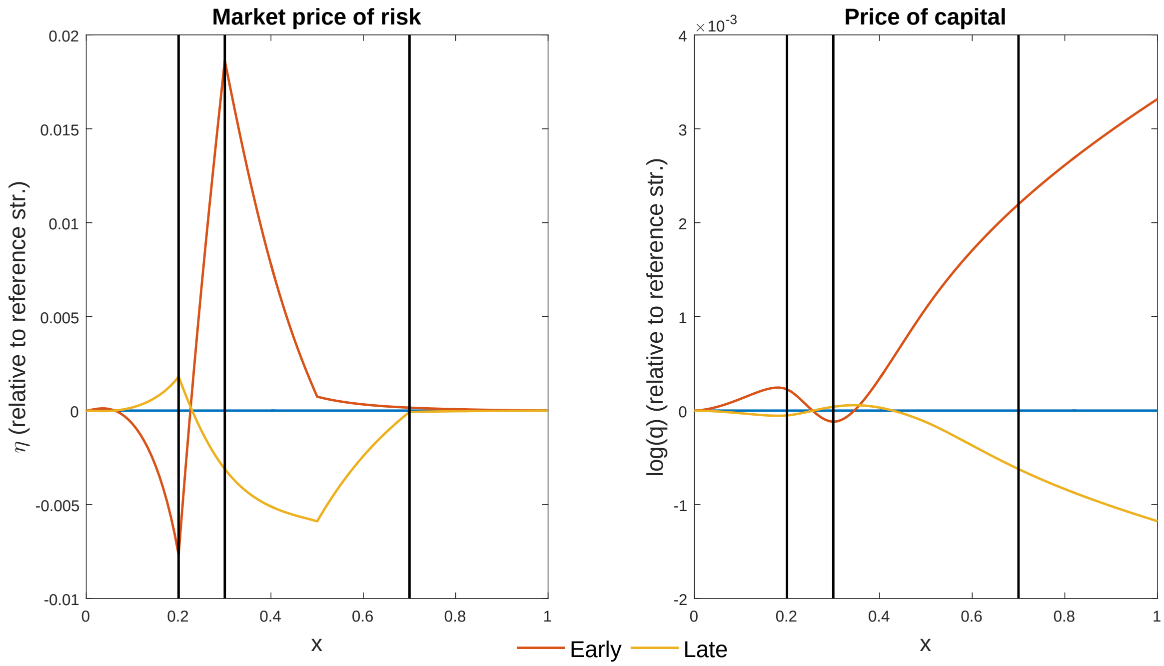
In the region the central bank is actually selling faster, for $0.2 \leq x \leq 0.3$, the market price of risk is increasing and it surpasses the risk premium under the reference strategy. As more assets are sold in the reference strategy in the region $0.3 \leq x \leq 0.5$, the difference between the early strategy and the reference strategy falls.

Figure 5: Policy Rules - Exit Strategies



This illustrates the fact that the effects of the central bank intervention on asset prices will depend in a subtle way on the incentives of financial intermediaries to take risk. By inducing the a hedging demand over the region $0 \leq x \leq 0.2$, the central bank is able to increase the price of capital, stimulating investment and growth. The same is true for higher levels of x. The reason is that now incentive to save does not fall as much as in the reference strategy. The only region where the early strategy obtains a smaller value of the price of capital is when the risky asset is effectively being sold.

Figure 6: Exit Strategies



Note: Red (orange) line indicates the difference between variable under early (late) exit and the reference strategy. Vertical lines indicates the point $x = 0.2$, $x = 0.3$, and $x = 0.7$. Solution evaluated at the ergodic mean of w_t.

7 Extensions

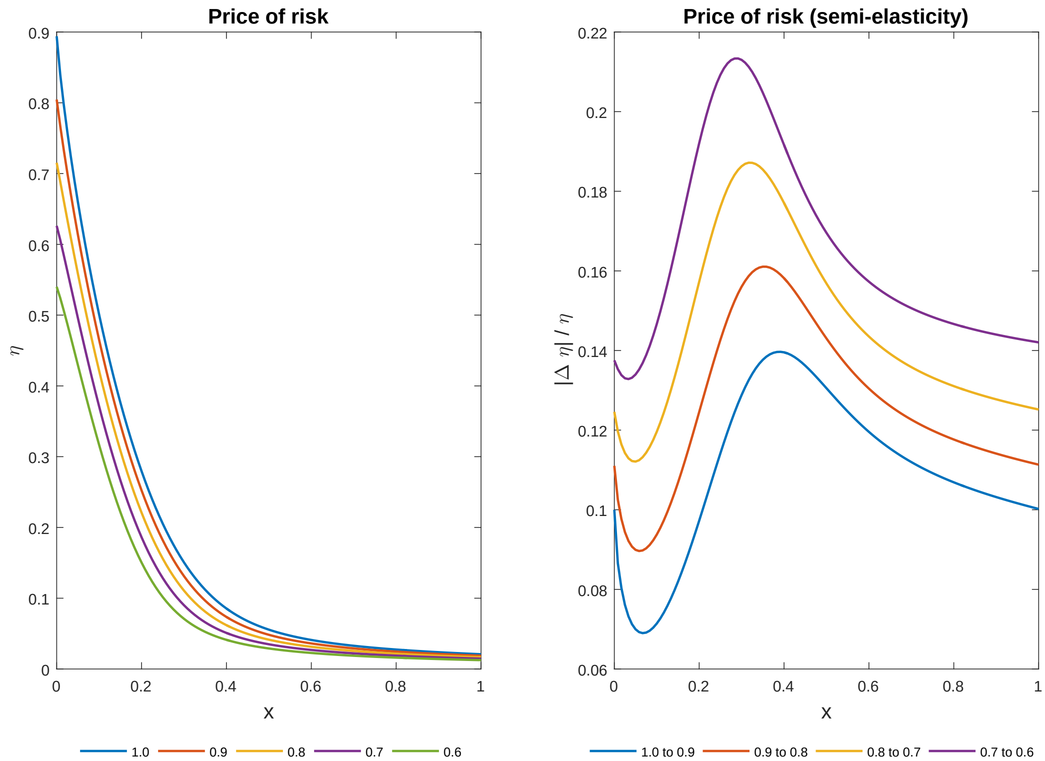
7.1 Effectiveness of Asset Purchases

We have seen in previous sections that asset purchases by the central bank affect the price of risk. However, the effect is not constant, in particular, it is state-dependent and non-linear. The effectiveness of unconventional monetary policy depends both on the strength of intermediaries' balance sheet and on the size of the intervention.

In order to isolate these effects, I will consider a simpler policy rule with $\beta_1^r(w) = 0$, so that policy does not respond to variations in x_t. Hence, the effect of the policy will vary with x_t only through the internal propagation mechanisms of the model.

The left panel on figure 7 shows the effect of assets purchases on the market price of risk for different levels of intervention. First, notice that effect gets smaller as intermediaries get better capitalized. Hence, unconventional monetary policy is more effective in crisis. The reason is that when savers are relatively more important, aggregate demand for risk becomes less elastic, so a larger change in returns is necessary to induce private agents to sell their risk assets to the central bank.

Figure 7: Effectiveness of Asset Purchases



Note: Left panel shows the price of risk for the values of $\omega^r \in \{0.6, 0.7, 0.8, 0.9, 1.0\}$. Right panel shows the proportional change in η for the specified changes in ω^r. Solution with the central bank evaluated at the ergodic mean of w_t.

The effect of asset purchases also depend on the size of the intervention. The right panel on figure 7 shows the semi-elasticity of the price of risk with respect to the net supply of risk for different levels of intervention.[34] Perhaps surprisingly, asset purchases become more effective as the size of the intervention increases. For small interventions, the level of endogenous volatility is relatively high, so the hedging effect is stronger. This attenuates the impact of the policy. For large interventions, endogenous volatility is low, so the hedging effect is weaker, and the policy becomes more powerful.

These two results have implications for the interpretation of the empirical evidence on the effects of QE. Researchers have found stronger effects for early interventions of the FED, exactly when intermediaries were less capitalized, consistent with the state-dependent effects described above.[35] However, the observation that the effectiveness of the policy increases with the size of the intervention indicates these estimates can be a poor guide of the potential impacts of unconventional monetary policy. The effect of large intervention can be significantly larger than captured by the initial estimates. The calibrated example indicates the effects of large policies can be up to 80% higher then the effects of small interventions.

[34]The graph can be read as follows: a reduction of 0.1 in the net asset supply when $x = 0.1$ will reduce the price of risk by about 8% (14%) the price of risk if we start at $\omega^r = 1.0$ ($\omega^r = 0.7$).

[35]See (Joyce et al., 2012) for a discussion of the empirical evidence.

7.2 Long-term bonds and term premium

The focus so far has been on the effects of asset purchases on the risk premium and the price of risky assets. Asset purchases also have implications for the price of long-term bonds and the term premium, even if the central bank does not buy long-term bonds directly.

Instead of considering the whole term structure and how it varies with the state variables, I will focus on the price of a long-term bond with exponentially decaying coupons $e^{-\delta_b t}$. This will provide a parsimonious way of capturing the responses of the term structure to the central bank policy.

The price of the bond, denoted by p_t, can be written as

$$p_t = \mathbb{E}_t\left[\int_t^\infty \frac{\pi_s}{\pi_t} e^{-\delta_b(s-t)} ds\right] = \int_t^\infty e^{-\delta_b(s-t)} p_{t,s} ds \tag{33}$$

where $p_{t,s} = \mathbb{E}_t\left[\frac{\pi_s}{\pi_t}\right]$ denotes the period t price of a zero-coupon bond maturing at date s.

The *yield* of the long-term bond is defined as the value of y_t satisfying

$$p_t = \int_t^\infty e^{-(\delta_b + y_t)(s-t)} ds \Rightarrow y_t = \frac{1}{p_t} - \delta_b$$

and similarly for zero-coupon bonds: $p_{t,s} = e^{-y_{t,s}(s-t)}$.

It can be shown that yield on the bond with decaying coupons is, up to first-order, an average of the yield on the zero-coupon bonds: [36]

$$y_t = \int_t^\infty (s-t)\delta_b^2 e^{-\delta_b(s-t)} y_{t,s} ds$$

We can define the term premium in a zero-coupon bond as:

$$\tau_{t,s} \equiv y_{t,s} - r^e_{t,s}$$

where $r^e_{t,s} = \frac{1}{s-t}\mathbb{E}\left[\int_t^s r_u du\right]$ is average expected interest rate between t and s.

The yield on the long-term bond can then be decomposed as an average of future expected short interest rates and an average term premium:

$$y_t = \underbrace{\int_t^\infty (s-t)\delta_b^2 e^{-\delta_b(s-t)} r^e_{t,s} ds}_{\equiv r^e_t} + \underbrace{\int_t^\infty (s-t)\delta_b^2 e^{-\delta_b(s-t)} \tau_{t,s} ds}_{\equiv \tau_t} \tag{34}$$

[36] To obtain the result, combine the expressions $p_t = \int_t^\infty e^{-\delta(s-t)} e^{-y_{t,s}(s-t)} ds \approx \frac{1}{\delta} - \int_t^\infty (s-t) e^{-\delta(s-t)} y_{t,s} ds$ and $p_t \approx \frac{1}{\delta_b} - \frac{1}{\delta_b^2} y_t$. Notice that $\delta^2 \int_t^\infty (s-t) e^{-\delta(s-t)} ds = 1$, so the weights integrate to one.

The next proposition shows how to obtain the price of the bond and the average expected interest rate by solving a partial differential equation:

Proposition 5. *The price of the bond $p_t = p(x_t, w_t)$ satisfy the condition:*

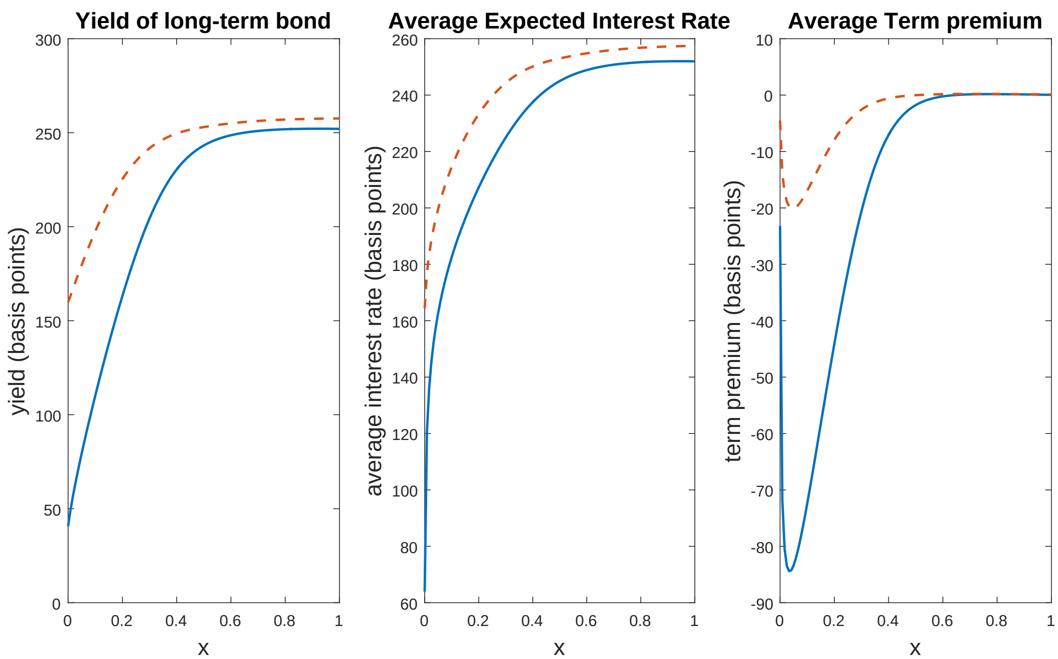
$$0 = \frac{\sigma_{x,t}^2}{2} p_{xx,t} + \sigma_{x,t}\sigma_{w,t} p_{xw,t} + \frac{\sigma_{w,t}^2}{2} p_{ww,t} + (\mu_{x,t} - \sigma_{x,t}\eta, t) p_{x,t} + (\mu_{w,t} - \sigma_{w,t}\eta, t) p_{w,t} + 1 - (r_t + \delta_b) p_t$$

The average expected interest rate $r_t^e = r^e(x_t, w_t)$ satisfy the condition:

$$0 = \frac{\sigma_{x,t}^2}{2} r_{xx,t}^e + \sigma_{x,t}\sigma_{w,t} r_{xw,t}^e + \frac{\sigma_{w,t}^2}{2} r_{ww,t}^e + \mu_{x,t} r_{x,t}^e + \mu_{w,t} r_{w,t}^e + \delta r_t - \delta r_t^e$$

Figure 8 shows the yield, the expected future interest rate, and the term premium as a function of the state variable x_t. The yield of the bond is increasing in x_t. Hence, the bond increases in value in bad times, providing an insurance against aggregate shocks. Given the insurance properties of bonds, the term premium is negative.[37]

Figure 8: Long-Term Bonds



Note: Solid (dashed) line represents laissez-faire (central bank) equilibrium. Solution with the central bank evaluated at the ergodic mean of w_t.

Purchases of risky assets *increases* the term premium. The intuition is simply that by reducing the amount of endogenous volatility in the economy, purchases of risky assets reduces the demand for insurance leading to an increase in the term premium. Given that the purchase of risky assets reduces

[37]Relatedly, the yield curve is downward-sloping, a common feature in models lacking inflation risk or (conventional) monetary shocks. One form of obtaining an upward-sloping yield curve would be to introduce preferences shocks to the model.

the risk premium, this result indicates there is a trade-off between risk and term premium: as the central bank reduces the risk premium, it increases the term premium.

8 Conclusion

In this paper, I studied the macroeconomic effects of large-scale asset purchases by central banks. I consider not only the immediate impact of the intervention, but also how expectation of future interventions affect asset prices and financial risk-taking. In line with the empirical evidence, I find that asset purchases reduces the risk premium and increase asset prices during crisis. In contrast, the expectation of future intervention have a negative impact on growth in normal times. As crisis gets less severe, investors have a weaker incentive to save, reducing growth. In contrast to what is typically argued in the popular press, I find that asset purchases reduce the concentration of risk on the hands of financial intermediaries. The reason is that intermediaries are more sensitive to variations in returns, so as returns fall with the intervention, they have a stronger incentive to sell than savers. I also consider the role of exit strategies. A commitment by the central bank to sell more of its assets in the future, conditional on the recovery of intermediaries' balance sheet, amplify the effects of asset purchases during crisis. By making the exit strategy conditional on the recovery, the central bank induces intermediaries to take more risk, reducing the risk premium.

This analysis suggests a few avenues for future research. First, a more detailed analysis of long-term bonds. This would require an extension with multiple shocks, as purchases of long-term bonds are redundant in the current setting. Multiple shocks would also allow for an analysis of transmission of asset purchases across different asset classes, as the increase in the exposure of the central bank to one risk factor may affect the premium for holding different risks. Second, the interaction between conventional and unconventional monetary policy. The results in this paper can be understood as the characterization of the natural (flexible price) allocation. In particular, my results show that the natural interest rate respond to the asset purchases of the central bank. This becomes particularly important under a binding zero lower bound, as an increase in the natural interest rates would stimulate the economy, as the interest gap would fall. Another important issue is the analysis of optimal policy. The presence of sticky prices would imply the central bank faces a trade-off: asset purchases stimulate the economy during crisis, but it distorts the allocation of risk to workers.

References

Adrian, Tobias, Hyun Song Shin et al., "Financial Intermediaries and Monetary Economics," *Handbook of Monetary Economics*, 2010, *3*, 601–650.

Allen, Franklin and Douglas Gale, "Limited market participation and volatility of asset prices," *The American Economic Review*, 1994, pp. 933–955.

Alvarez, Fernando, Andrew Atkeson, and Patrick J Kehoe, "Money, Interest Rates, and Exchange Rates with Endogenously Segmented Markets," *Journal of Political Economy*, 2002, *110* (1).

Andrés, Javier, J David López-Salido, and Edward Nelson, "Tobin's Imperfect Asset Substitution in Optimizing General Equilibrium.," *Journal of Money, Credit & Banking (Ohio State University Press)*, 2004, *36* (4).

Araújo, Aloísio, Susan Schommer, and Michael Woodford, "Conventional and Unconventional Monetary Policy with Endogenous Collateral Constraints," *American Economic Journal: Macroeconomics*, 2015, *7* (1), 1–43.

Barro, Robert J and Andrew Mollerus, "Safe Assets," Technical Report, National Bureau of Economic Research 2014.

Basak, Suleyman and Domenico Cuoco, "An equilibrium model with restricted stock market participation," *Review of Financial Studies*, 1998, *11* (2), 309–341.

Begenau, Juliane, Monika Piazzesi, and Martin Schneider, "Banks' Risk Exposures," 2013.

Bernanke, Ben S, Mark Gertler, and Simon Gilchrist, "The financial accelerator in a quantitative business cycle framework," *Handbook of macroeconomics*, 1999, *1*, 1341–1393.

Berriel, Tiago C and Arthur Mendes, "Central Bank Balance Sheet, Liquidity Trap, and Quantitative Easing," Technical Report, working paper 2015.

Bhatarai, Saroj, Bulat Gafarov, and Gauti Eggertsson, "Time Consistency and the Duration of Government Debt: A Signalling Theory of Quantitative Easing," in "2014 Meeting Papers" number 1292 Society for Economic Dynamics 2014.

Brav, Alon, George M Constantinides, and Christopher C Geczy, "Asset Pricing with Heterogeneous Consumers and Limited Participation: Empirical Evidence," *Journal of Political Economy*, 2002, *110* (4).

Brunner, Karl and Allan H Meltzer, "Mr. Hicks and the" Monetarists"," *Economica*, 1973, pp. 44–59.

Brunnermeier, Markus and Yuliy Sannikov, "A Macroeconomic Model with a Financial Sector," *American Economic Review*, 2014.

Brunnermeier, Markus K. and Yuliy Sannikov, "The I Theory of Money," 2015.

Caballero, Ricardo J and Emmanuel Farhi, “The safety trap,” Technical Report, National Bureau of Economic Research 2014.

Chan, Yeung Lewis and Leonid Kogan, “Catching Up with the Joneses: Heterogeneous Preferences and the Dynamics of Asset Prices,” *Journal of Political Economy*, 2002, *110* (6), 1255–1285.

Chen, Han, Vasco Cúrdia, and Andrea Ferrero, “The Macroeconomic Effects of Large-scale Asset Purchase Programmes*,” *The economic journal*, 2012, *122* (564), F289–F315.

Chien, Yili, Harold Cole, and Hanno Lustig, “Is the Volatility of the Market Price of Risk Due to Intermittent Portfolio Rebalancing?,” *The American Economic Review*, 2012, *102* (6), 2859–2896.

Cochrane, John H, “Presidential address: Discount rates,” *The Journal of Finance*, 2011, *66* (4), 1047–1108.

Constantinides, George M, “Intertemporal asset pricing with heterogeneous consumers and without demand aggregation,” *Journal of Business*, 1982, pp. 253–267.

Curdia, Vasco and Michael Woodford, “The central-bank balance sheet as an instrument of monetary-policy,” *Journal of Monetary Economics*, 2011, *58* (1), 54–79.

D’Amico, Stefania and Thomas B King, “Flow and stock effects of large-scale treasury purchases: Evidence on the importance of local supply.,” *Journal of financial economics*, 2013, *108* (2), 425–448.

Diamond, Douglas W and Raghuram G Rajan, “Illiquid Banks, Financial Stability, and Interest Rate Policy,” *Journal of Political Economy*, 2012, *120* (3), 552–591.

Drechsler, Itamar, Alexi Savov, and Philipp Schnabl, “A model of monetary policy and risk premia,” Technical Report, National Bureau of Economic Research 2014.

Duffie, Darrell and Larry G Epstein, “Stochastic differential utility,” *Econometrica: Journal of the Econometric Society*, 1992, pp. 353–394.

Dumas, Bernard, “Two-person dynamic equilibrium in the capital market,” *Review of Financial Studies*, 1989, *2* (2), 157–188.

Eggertsson, Gauti B and Michael Woodford, “Zero bound on interest rates and optimal monetary policy,” *Brookings Papers on Economic Activity*, 2003, *2003* (1), 139–233.

Fawley, Brett W and Christopher J Neely, “Four stories of quantitative easing,” *Federal Reserve Bank of St. Louis Review*, 2013, *95* (January/February 2013).

Gabaix, Xavier, “A sparsity-based model of bounded rationality,” *The Quarterly Journal of Economics*, 2014, *129* (4), 1661–1710.

Gagnon, Joseph, Matthew Raskin, Julie Remache, and Brian Sack, “The financial market effects of the Federal Reserve’s large-scale asset purchases,” *International Journal of Central Banking*, 2011, *7* (1), 3–43.

Gârleanu, Nicolae and Lasse Heje Pedersen, "Margin-based asset pricing and deviations from the law of one price," *Review of Financial Studies*, 2011, p. hhr027.

__ **and Stavros Panageas**, "Young, Old, Conservative, and Bold: The Implications of Heterogeneity and Finite Lives for Asset Pricing," *Journal of Political Economy*, 2015, *123* (3), 670–685.

Gennaioli, Nicola, Andrei Shleifer, and Robert Vishny, "Neglected risks, financial innovation, and financial fragility," *Journal of Financial Economics*, 2012, *104* (3), 452–468.

Gertler, Mark and Nobuhiro Kiyotaki, "Financial intermediation and credit policy in business cycle analysis," *Handbook of monetary economics*, 2010, *3* (3), 547–599.

__ **and Peter Karadi**, "A model of unconventional monetary policy," *Journal of monetary Economics*, 2011, *58* (1), 17–34.

__ **and** __ , "Qe 1 vs. 2 vs. 3...: A framework for analyzing large-scale asset purchases as a monetary policy tool," *International Journal of Central Banking*, 2013, *9* (1), 5–53.

__ **, Nobuhiro Kiyotaki, and Andrea Prespitino**, "Wholesale Banking and Bank Runs in Macroeconomic Modelling of Financial Crises," 2015.

Gomes, Francisco and Alexander Michaelides, "Optimal Life-Cycle Asset Allocation: Understanding the Empirical Evidence," *The Journal of Finance*, 2005, *60* (2), 869–904.

Greenwood, Robin and Dimitri Vayanos, "Bond supply and excess bond returns," *Review of Financial Studies*, 2014, *27* (3), 663–713.

Grossman, Sanford and Laurence Weiss, "A Transactions-Based Model of the Monetary Transmission Mechanism," *The American Economic Review*, 1983, *73* (5), 871–880.

Gruber, Jonathan, "A Tax-Based Estimate of the Elasticity of Intertemporal Substitution.," *Quarterly Journal of Finance*, 2013, *3* (1), 1 – 20.

Gurley, John G, Edward Stone Shaw, and Alain C Enthoven, *Money in a Theory of Finance*, Brookings Institution Washington, DC, 1960.

Guvenen, Fatih, "A parsimonious macroeconomic model for asset pricing," *Econometrica*, 2009, *77* (6), 1711–1750.

Hall, Robert E. and Ricardo Reis, "Maintaining Central-Bank Solvency under New-Style Central Banking," February 2013. Hoover Institution, Stanford University, and Columbia University.

Hayashi, Fumio, "Tobin's marginal q and average q: A neoclassical interpretation," *Econometrica: Journal of the Econometric Society*, 1982, pp. 213–224.

He, Zhigu and Arvind Krishnamurthy, "A model of capital and crises," *The Review of Economic Studies*, 2012, *79* (2), 735–777.

He, Zhiguo and Arvind Krishnamurthy, "Intermediary Asset Pricing," *The American Economic Review*, 2013, *103* (2), 732–70.

Holmstrom, Bengt and Jean Tirole, "Financial intermediation, loanable funds, and the real sector," *the Quarterly Journal of economics*, 1997, pp. 663–691.

Joyce, Michael, David Miles, Andrew Scott, and Dimitri Vayanos, "Quantitative Easing and Unconventional Monetary Policy–an Introduction*," *The Economic Journal*, 2012, *122* (564), F271–F288.

Justiniano, Alejandro, Giorgio E Primiceri, and Andrea Tambalotti, "Investment shocks and business cycles," *Journal of Monetary Economics*, 2010, *57* (2), 132–145.

Kapoor, Mudit and Shamika Ravi, "Elasticity of Intertemporal Substitution in Consumption: Empirical Evidence from a Natural Experiment," *SSRN Working Paper Series*, 2010.

Kiyotaki, Nobuhiro and John Moore, "Credit Cycles," *The Journal of Political Economy*, 1997, *105* (2), 211–248.

Krishnamurthy, Arvind and Annette Vissing-Jorgensen, "The Effects of Quantitative Easing on Interest Rates: Channels and Implications for Policy," *Brookings Papers on Economic Activity*, 2011, *2011* (2), 215–287.

__ **and** __ , "The aggregate demand for treasury debt," *Journal of Political Economy*, 2012, *120* (2), 233–267.

Longstaff, Francis A and Jiang Wang, "Asset pricing and the credit market," *Review of Financial Studies*, 2012, *25* (11), 3169–3215.

Mankiw, N Gregory and Stephen P Zeldes, "The consumption of stockholders and nonstockholders," *Journal of Financial Economics*, 1991, *29* (1), 97–112.

Negro, Marco Del, Gauti Eggertsson, Andrea Ferrero, and Nobuhiro Kiyotaki, "The Great Escape? A Quantitative Evaluation of the Fed's Liquidity Facilities," *Federal Reserve Bank of New York Staff Report*, 2011, (520).

Reis, Ricardo, "Different Types of Central Bank Insolvency and the Central Role of Seignorage," *Journal of Monetary Economics*, 2015, *73*, 20 – 25. Carnegie-Rochester-NYU Conference Series on Public Policy "Monetary Policy: An Unprecedented Predicament" held at the Tepper School of Business, Carnegie Mellon University, November 14-15, 2014.

Rotemberg, Julio J, "A Monetary Equilibrium Model with Transactions Costs," *The Journal of Political Economy*, 1984, *92* (1), 40–58.

Schiesser, WE, "PDE boundary conditions from minimum reduction of the PDE," *Applied numerical mathematics*, 1996, *20* (1), 171–179.

Silva, Dejanir, "Optimal Unconventional Monetary Policy: A Two-Period Example," Technical Report, mimeo, MIT 2015.

Sims, Christopher A, "Implications of rational inattention," *Journal of monetary Economics*, 2003, *50* (3), 665–690.

Tella, Sebastian Di, "Uncertainty Shocks and Balance Sheet Recessions," Technical Report, mimeo, MIT 2012.

Tobin, James, "A general equilibrium approach to monetary theory," *Journal of money, credit and banking*, 1969, *1* (1), 15–29.

_ **and William C Brainard**, "Financial intermediaries and the effectiveness of monetary controls," *The American Economic Review*, 1963, pp. 383–400.

Vayanos, Dimitri and Jean-Luc Vila, "A preferred-habitat model of the term structure of interest rates," Technical Report, National Bureau of Economic Research 2009.

Vissing-Jørgensen, Annette and Orazio P Attanasio, "Stock-market participation, intertemporal substitution, and risk-aversion," *American Economic Review*, 2003, pp. 383–391.

Wallace, Neil, "A Modigliani-Miller theorem for open-market operations," *The American Economic Review*, 1981, pp. 267–274.

Wang, Jiang, "The term structure of interest rates in a pure exchange economy with heterogeneous investors," *Journal of Financial Economics*, 1996, *41* (1), 75–110.

Williamson, Stephen D, "Liquidity, monetary policy, and the financial crisis: A new monetarist approach," *The American Economic Review*, 2012, pp. 2570–2605.

A Proofs

A.1 Proof of proposition 1

Proof. The assumptions on the central bank imply $w_t = 0$ for all $t \geq 0$. Since $\gamma_b = \gamma_s$, then $\sigma_{x,t} = 0$, what imply $\sigma_{\zeta,t} = \sigma_{\xi,t} = \sigma_{q,t} = 0$. Hence, up to scale, the economy is non-stochastic. The assumption on the initial net worth imply that $\mu_{x,t} = 0$.

The market price of risk is given by $\eta_t = \gamma\sigma$. Risk exposures are given by $\sigma_{b,t} = \sigma_{s,t} = \sigma$. Combining the market clearing condition for consumption and the pricing condition for capital, we obtain:

$$r_t = \rho + \psi^{-1} g_t - \left(1 + \psi^{-1}\right) \frac{\gamma\sigma^2}{2}$$

Plugging the expression above into the pricing condition for capital:

$$\frac{A - \iota(g_t)}{q_t} = \rho - \left(1 - \psi^{-1}\right)\left(g_t - \frac{\gamma\sigma^2}{2}\right)$$

The condition above combined with $\iota'(g_t) = q_t$ determine g_t and q_t. Consider the quadratic adjustment costs case:

$$\iota(g) = \phi_0(g + \delta) + \frac{\phi_1}{2}(g + \delta)^2$$

Growth rate is given by $g_t = \frac{q - \phi_0}{\phi_1} - \delta$. Plugging into the expression above, we obtain $\iota(g(q)) = \frac{q^2 - \phi_0^2}{2\phi_1}$. The pricing condition for capital can then be written as

$$(1 - 2\psi^{-1})q_t^2 - 2\phi_1\left[\rho + \left(1 - \psi^{-1}\right)\left(\frac{\phi_0}{\phi_1} + \delta + \frac{\gamma\sigma^2}{2}\right)\right] q_t + 2\phi_1 A + \phi_0^2 = 0$$

It is instructive to consider first the limit $\phi_1 \to \infty$. In this case $g_t = -\delta$. Price is given by

$$q_t = \frac{A}{\rho - (1 - \psi^{-1})\left(g_t - \frac{\gamma\sigma^2}{2}\right)}$$

and the existence of a positive price requires

$$\rho - (1 - \psi^{-1})\left(g_t - \frac{\gamma\sigma^2}{2}\right) > 0$$

This is the usual condition for existence of equilibrium in a Lucas tree model with Epstein-Zin preferences. For the case with finite ϕ_1 the condition is given by

$$\rho - (1 - \psi^{-1})\left(\overline{g} - \frac{\gamma\sigma^2}{2}\right) > 0 \tag{A.1}$$

where

$$\overline{g} \equiv \frac{\sqrt{2\phi_1 A + \phi_0^2} - \phi_0}{\phi_1} - \delta$$

Let's now check this is indeed the case. The consumption-wealth ratio will be positive if

$$\phi_1 \left[\rho + \left(1 - \psi^{-1}\right)\left(\frac{\phi_0}{\phi_1} + \delta + \frac{\gamma\sigma^2}{2}\right)\right] - (1 - \psi^{-1})q_t > 0 \tag{A.2}$$

If $\psi = 2$, then the price is given by

$$q_t = \frac{A + \frac{\phi_0^2}{2\phi_1}}{\rho + \frac{1}{2}\left(\frac{\phi_0}{\phi_1} + \delta + \frac{\gamma\sigma^2}{2}\right)}$$

Plugging the expression above into (A.2), we obtain (A.1) for $\psi = 2$.

Define the following coefficients:

$$B = \phi_1 \left[\rho + \left(1 - \psi^{-1}\right)\left(\frac{\phi_0}{\phi_1} + \delta + \frac{\gamma\sigma^2}{2}\right)\right]; \qquad C = 2\phi_1 A + \phi_0^2;$$

If $\psi < 2$, then the quadratic equation for p_t has a unique positive root:

$$q_t = \frac{\sqrt{B^2 + (2\psi^{-1} - 1)C} - B}{2\psi^{-1} - 1}$$

Condition (A.2) can be written as

$$B + (1 - \psi)\sqrt{B^2 + (2\psi^{-1} - 1)C} > 0 \iff B - (1 - \psi^{-1})\sqrt{C} > 0$$

and the second inequality is equivalent to (A.1).

In order to see the equivalence between the two inequalities, notice that for a given C, the two functions of B at the left-hand side are strictly increasing and have a zero at the same point, so the two functions are positive for the same set of values of B (given C).

If $\psi > 2$ and the quadratic equation has real roots, then there is a single root consistent with positive consumption-wealth ratio:

$$q_t = \frac{B - \sqrt{B^2 - (1 - 2\psi^{-1})C}}{1 - 2\psi^{-1}}$$

The consumption-wealth ratio will be positive if

$$B - (\psi - 1)\sqrt{B^2 - (1 - 2\psi^{-1})C} < 0 \text{ and } B^2 - (1 - 2\psi^{-1})C \geq 0 \iff B - (1 - \psi^{-1})\sqrt{C} > 0$$

and the second inequality is equivalent to (A.1).

A similar argument applies: for the region where the term inside the square root is non-negative, the function on the left of the first inequality is decreasing in B and it is equal to zero and the function on the left of the last inequality is zero. Hence, the two sets of inequalities are equivalent.

□

A.2 Proof of proposition 2

Proof. First, notice the market price of risk can be written as

$$\eta_t = \gamma_b \sigma_{b,t} - (1-\gamma_b)\frac{\zeta_{x,t}}{\zeta_t}\sigma_{x,t}$$

Using the market clearing condition for risk, we obtain

$$x_t\sigma_{b,t} + (1-x_t)\frac{1}{\gamma_s}\left[\gamma_b\sigma_{b,t} + \left((1-\gamma_s)\frac{\xi_{x,t}}{\xi_t} - (1-\gamma_b)\frac{\zeta_{x,t}}{\zeta_t}\right)\sigma_{x,t}\right] = \omega_t^r(\sigma+\sigma_{q,t})$$

Using the market clearing condition for risk and (30), we can write

$$\sigma_{x,t} = x_t\left[\sigma_{b,t} - \omega_t^r(\sigma+\sigma_{q,t})\right]$$

Combining the previous two expressions, we get

$$x_t\tilde{\sigma}_{b,t} + (1-x_t)\frac{1}{\gamma_s}\left[\gamma_b\tilde{\sigma}_{b,t} + \left((1-\gamma_s)\frac{\xi_{x,t}}{\xi_t} - (1-\gamma_b)\frac{\zeta_{x,t}}{\zeta_t}\right)x_t\left[\tilde{\sigma}_{b,t}-1\right]\right] = 1$$

where $\tilde{\sigma}_{b,t} \equiv \frac{\sigma_{b,t}}{\omega_t^r(\sigma+\sigma_{q,t})}$.

Rearranging the expression the above,

$$\tilde{\sigma}_{b,t} = \frac{1+\frac{x_t(1-x_t)}{\gamma_s}\left((1-\gamma_s)\frac{\xi_{x,t}}{\xi_t} - (1-\gamma_b)\frac{\zeta_{x,t}}{\zeta_t}\right)}{1+\frac{x_t(1-x_t)}{\gamma_s}\left((1-\gamma_s)\frac{\xi_{x,t}}{\xi_t} - (1-\gamma_b)\frac{\zeta_{x,t}}{\zeta_t}\right) - \frac{\gamma_s-\gamma_b}{\gamma_s}(1-x_t)} \tag{A.3}$$

Notice that $\tilde{\sigma}_{b,t}$ is increasing in $\frac{\gamma_s-\gamma_b}{\gamma_s}\frac{1-x_t}{1+\mathcal{H}_t}$, where $\mathcal{H}_t \equiv \frac{x_t(1-x_t)}{\gamma_s}\left((1-\gamma_s)\frac{\xi_{x,t}}{\xi_t} - (1-\gamma_b)\frac{\zeta_{x,t}}{\zeta_t}\right)$. Let's compute how this term responds to γ_s:

$$\left.\frac{\partial\left[\frac{\gamma_s-\gamma_b}{\gamma_s}\frac{1-x_t}{1+\mathcal{H}_t}\right]}{\partial\gamma_s}\right|_{\gamma_s=\gamma_b} = \frac{1-x_t}{\gamma_b} \tag{A.4}$$

where I used the fact that $\mathcal{H}_t = 0$ for $\gamma_s = \gamma_b$.

Since $\tilde{\sigma}_{b,t} = 1$ for $\gamma_s = \gamma_b$ and it is (locally) increasing in γ_s, then $\tilde{\sigma}_{b,t} > 1$ for $\gamma_s = \gamma_b + \epsilon$, $\epsilon > 0$

sufficiently small. Market clearing imply $\tilde{\sigma}_{s,t} < 1$, so $\sigma_{b,t} > \sigma_{s,t}$.[38] □

A.3 Proof of proposition 3

Define $\mathcal{T}_{j,t}$ as the expected discounted value of transfers:

$$\mathcal{T}_{j,t} \equiv \mathbb{E}\left[\int_t^{\infty} \frac{\pi_s}{\pi_t} T_{j,s} ds\right]$$

Define the martingale process $G_{j,t}$:

$$G_{j,t} = \mathbb{E}_t\left[\int_0^{\infty} \frac{\pi_s}{\pi_0} T_{j,s} ds\right] = \int_0^t \frac{\pi_s}{\pi_0} T_{j,s} ds + \frac{\pi_t}{\pi_0}\mathcal{T}_{j,t}$$

The martingale representation theorem implies that there exists a process $\sigma_{G_j,t}$ such that

$$dG_{j,t} = \frac{\pi_t}{\pi_0}\sigma_{G_j,t} dZ_t$$

Combining the previous two expressions, we get

$$T_{j,t}dt + \frac{d\left(\pi_t \mathcal{T}_{j,t}\right)}{\pi_t} = \sigma_{G_j,t} dZ_t$$

Applying Ito's lemma, we obtain

$$\mu_{\mathcal{T}_j,t} = r_t \mathcal{T}_{j,t} + \eta_t \sigma_{\mathcal{T}_j,t} - T_{j,t}$$

where $\sigma_{H_j,t} = \sigma_{G_j,t} + \mathcal{T}_{j,t}\eta_t$.

Define total wealth as the sum of financial wealth and the value of transfers:

$$\tilde{n}_{j,t} = n_{j,t} + \mathcal{T}_{j,t}$$

which evolves according to

$$\frac{d\tilde{n}_{j,t}}{\tilde{n}_{j,t}} = \left[r_t + \tilde{\sigma}_{j,t}\eta_t - \tilde{c}_{j,t}\right] dt + \tilde{\sigma}_{j,t} dZ_t \tag{A.5}$$

where

$$\tilde{\sigma}_{j,t} = \frac{n_{j,t}}{\tilde{n}_{j,t}}\sigma_{j,t} + \frac{\sigma_{\mathcal{T}_j,t}}{\tilde{n}_j, t}; \qquad \tilde{c}_{j,t} = \frac{c_{j,t}}{\tilde{n}_{j,t}}$$

The problem of investor j can alternatively be written as choosing $(c_j, \tilde{\sigma}_j)$ subject to (A.5), given $\tilde{n}_{j,0} > 0$.

Fix a set of policy rules $\sigma_{cb,t}$ and $T_{j,t}$ and the corresponding equilibrium prices (r_t, η_t, S_t). Consider an alternative set of policy rules $\sigma^*_{cb,t}$ and $T^*_{j,t}$ such that total wealth at period 0 is unchanged for both

[38] $\tilde{\sigma}_{b,t}$ is locally increasing for $x_t < 1$, but we can check directly that $\sigma_{b,t} > \sigma_{s,y}$ as x_t approaches.

types when computed using the state price density from the *initial* equilibrium. I will conjecture that prices, investment, and consumption are unchanged $(r_t^*, \eta_t^*, \pi_t^*, S_t^*, g_t^*, c_t^*) = (r_t, \eta_t, \pi_t, S_t, g_t, c_t)$ and risk exposures are given by

$$n_{j,t}^* \sigma_{j,t}^* = n_{j,t}\sigma_{j,t} - \left(\sigma_{\mathcal{T}_j,t}^* - \sigma_{\mathcal{T}_j,t}\right) \tag{A.6}$$

Let's guess and verify this is an equilibrium. By assumption $\tilde{n}_{j,0}^* = \tilde{n}_{j,t}$. The budget set for investor j is unchanged, then $(\tilde{\sigma}_j^*, c_j^*) = (\tilde{\sigma}_j, c_j)$ and equation (A.6) hold. Since the state price density is the same, $g^* = g$. Consumption did not change, so the market clearing condition for consumption hold. It remains to determine the remainder market clearing conditions hold.

The budget of the central bank can be written as

$$n_{cb,t} = \mathbb{E}_t\left[\int_t^\infty \frac{\pi_s}{\pi_t} T_s ds\right] = \mathcal{T}_{b,t} + \mathcal{T}_{s,t}$$

This imply the following chain of equalities

$$n_{b,t}^* + n_{s,t}^* + n_{cb,t}^* = \tilde{n}_{b,t}^* + \tilde{n}_{s,t}^* = \tilde{n}_{b,t} + \tilde{n}_{s,t} = n_{b,t} + n_{s,t} + n_{cb,t} = S_t$$

where I used the fact that total wealth did not change with the new policy.

Since $n_{cb,t} = \mathcal{T}_{b,t} + \mathcal{T}_{s,t}$, we have that

$$\sigma_{cb,t} n_{cb,t} = \sigma_{\mathcal{T}_b,t} + \sigma_{\mathcal{T}s,t}$$

We can use this fact to show the remaining market clearing condition hold:

$$\sigma_{b,t}^* n_{b,t}^* + \sigma_{s,t}^* n_{s,t}^* + \sigma_{cb,t}^* n_{cb,t}^* = \tilde{\sigma}_{b,t}^* \tilde{n}_{b,t}^* + \tilde{\sigma}_{s,t}^* \tilde{n}_{s,t}^* = \tilde{\sigma}_{b,t}\tilde{n}_{b,t} + \tilde{\sigma}_{s,t}\tilde{n}_{s,t} = \sigma_{b,t} n_{b,t} + \sigma_{s,t} n_{s,t} + \sigma_{cb,t} n_{cb,t} = \sigma_{S,t} S_t$$

This concludes the proof that the conjectured allocation is an equilibrium. □

A.4 Proof of proposition 4

Proof. Law of motion of X_t: Applying Ito's lemma to the definition of x_t and w_t we obtain:

$$
\begin{aligned}
dx_t &= x_t \left[\frac{dn_{b,t}}{n_{b,t}} - \frac{d(n_{b,t}+n_{s,t})}{n_{b,t}+n_{s,t}} + \left(\frac{d(n_{b,t}+n_{s,t})}{n_{b,t}+n_{s,t}} \right)^2 - \frac{d(n_{b,t}+n_{s,t})}{n_{b,t}+n_{s,t}} \frac{dn_{b,t}}{n_{b,t}} \right] \\
&= \underbrace{x_t(1-x_t)\left[(\sigma_{b,t}-\sigma_{s,t})\left(\eta_t - \omega_t^r(\sigma+\sigma_{q,t})\right) + \hat{c}_{s,t} - \hat{c}_{b,t} \right] - \kappa\left(x_t - \theta_b\right)}_{\mu_{x,t}} dt + \\
&+ \underbrace{x_t(1-x_t)(\sigma_{b,t}-\sigma_{s,t})}_{\sigma_{x,t}} dZ_t \\
dw_t &= w_t \left[\frac{dn_{cb,t}}{n_{cb,t}} - \frac{d(q_t Y_t)}{q_t Y_t} + \frac{d(q_t K_t)^2}{(q_t K_t)^2} - \frac{dn_{cb,t}}{n_{cb,t}} \frac{d(q_t K_t)}{q_t K_t} \right] \\
&= \underbrace{w_t\left[r_t + \sigma_{cb,t}\eta_t - \hat{T}_t - \left(\mu_{q,t} + g_t + \sigma\sigma_{p,t}\right) \right] - (1-w_t)(1-\omega_t^r)(\sigma+\sigma_{p,t})^2}_{\mu_{w,t}} dt + \\
&+ \underbrace{(1-w_t)(1-\omega_t^r)(\sigma+\sigma_{q,t})}_{\sigma_{w,t}} dZ_t
\end{aligned}
$$

using the fact $\sigma_{cb,t} = \frac{1-\omega_t^r(1-w_t)}{w_t}(\sigma+\sigma_{q,t})$.

Let's solve for the drift and diffusion terms as a function of the net worth multipliers and their derivatives.

Diffusion: The diffusion of X_t is given by

$$
\sigma_{x,t} = x_t(1-x_t)\left(\sigma_{b,t}-\sigma_{s,t}\right); \qquad \sigma_{w,t} = (1-w_t)(1-\omega_t^r)(\sigma+\sigma_{q,t});
$$

The risk exposure of intermediaries and savers can be written as a function of $\sigma_{x,t}$:

$$
\sigma_{s,t} = \frac{\eta_t}{\gamma_s} - \frac{1}{\gamma_s}\left(\hat{\xi}_x \sigma_{x,t} + \hat{\xi}_w \sigma_{w,t}\right); \qquad \sigma_{b,t} = \frac{\eta_t}{\gamma_b} - \frac{1}{\gamma_b}\left(\hat{\zeta}_x \sigma_{x,t} + \hat{\zeta}_w \sigma_{w,t}\right); \tag{A.7}
$$

where $\hat{\xi}_t = (\gamma_s - 1)\log \xi_t$ and $\hat{\zeta}_t = (\gamma_b - 1)\log \zeta_t$ (for instance, $\hat{\xi}_x = (\gamma_s - 1)\frac{\xi_x}{\xi}$).

Plugging in the expressions for $(\sigma_{\xi,t}, \sigma_{\zeta,t})$ into the market clearing condition for capital, we obtain:

$$
\eta_t = \gamma_t \left[\omega_t^r(\sigma+\sigma_{q,t}) + \frac{x_t}{\gamma_b}\left(\hat{\zeta}_x \sigma_{x,t} + \hat{\zeta}_w \sigma_{w,t}\right) + \frac{1-x_t}{\gamma_s}\left(\hat{\xi}_x \sigma_{x,t} + \hat{\xi}_w \sigma_{w,t}\right) \right] \tag{A.8}
$$

The risk exposures can then be written as

$$
\begin{aligned}
\sigma_{b,t} &= \frac{\gamma_t}{\gamma_b}\left[\omega_t(\sigma+\sigma_{q,t}) + \frac{1-x_t}{\gamma_s}\left[\left(\hat{\xi}_x-\hat{\zeta}_x\right)\sigma_{x,t} + \left(\hat{\xi}_w-\hat{\zeta}_w\right)\sigma_{w,t}\right]\right] \\
\sigma_{s,t} &= \frac{\gamma_t}{\gamma_s}\left[\omega_t(\sigma+\sigma_{q,t}) - \frac{x_t}{\gamma_b}\left[\left(\hat{\xi}_x-\hat{\zeta}_x\right)\sigma_{x,t} + \left(\hat{\xi}_w-\hat{\zeta}_w\right)\sigma_{w,t}\right]\right]
\end{aligned}
$$

Plugging in the expression above into the equation for $\sigma_{x,t}$, we get

$$
\begin{aligned}
\sigma_{x,t} &= x_t(1-x_t)\frac{\gamma_t}{\gamma_b\gamma_s}\left[(\gamma_s-\gamma_b)\omega_t^r(\sigma+\sigma_{q,t}) + \left(\hat{\xi}_x-\hat{\zeta}_x\right)\sigma_{x,t} + \left(\hat{\xi}_w-\hat{\zeta}_w\right)\sigma_{w,t}\right] \\
\sigma_{w,t} &= w_t(l_{cb,t}-1)\left(\sigma+\sigma_{q,t}\right)
\end{aligned}
$$

Consider first the special case where there is no intervention, i.e., $w_t = 0$. In this case, we can easily solve for $\sigma_{x,t}$:

$$
\sigma_{x,t} = \frac{\gamma_t\frac{x_t}{\gamma_b}\frac{1-x_t}{\gamma_s}(\gamma_s-\gamma_b)}{1-\gamma_t\frac{x_t}{\gamma_b}\frac{1-x_t}{\gamma_s}\left[(\gamma_s-\gamma_b)\frac{q_x}{q} + \left(\hat{\xi}_x-\hat{\zeta}_x\right)\right]}\sigma \tag{A.9}
$$

Let's go back to the general case. The diffusion of $w_{cb,t}$ can be written as

$$
\sigma_{w,t} = \frac{w_{cb,t}(l_{cb,t}-1)\left(\sigma+\frac{q_x}{q}\sigma_{x,t}\right)}{1-w_{cb,t}(l_{cb,t}-1)\frac{q_w}{q}} \tag{A.10}
$$

and $\sigma+\sigma_{p,t}$ is given by

$$
\sigma+\sigma_{q,t} = \frac{\sigma+\frac{q_x}{q}\sigma_{x,t}}{1-w_t(l_{cb,t}-1)\frac{p_w}{p}} \tag{A.11}
$$

The diffusion of x_b can be written as

$$
\sigma_{x,t} = \frac{\gamma_t\frac{x_t}{\gamma_b}\frac{1-x_t}{\gamma_s}\left[\frac{(\gamma_s-\gamma_b)\omega_t+w_t(l_{cb,t}-1)(\hat{\xi}_w-\hat{\zeta}_w)}{1-w_t(l_{cb,t}-1)\frac{q_w}{q}}\right]}{1-\gamma_t\frac{x_t}{\gamma_b}\frac{1-x_t}{\gamma_s}\left[\frac{(\gamma_s-\gamma_b)\omega_t\frac{q_x}{q}+w_t(l_{cb,t}-1)\frac{q_x}{q}\left(\hat{\xi}_w-\hat{\zeta}_w\right)}{1-w_t(l_{cb,t}-1)\frac{p_w}{p}} + \left(\hat{\xi}_x-\hat{\zeta}_x\right)\right]}\sigma \tag{A.12}
$$

The expression above depends on the derivatives of the net worth multipliers, but it depends on the derivatives of the price-output ratio as well. However, we can use the market clearing condition for goods to eliminate the derivatives involving p. The following expressions show how to obtain p and its derivatives from the net worth multipliers (and their derivatives):

$$
\begin{aligned}
\frac{\alpha(x,w)}{q(x,w)} &= x\rho^{\psi}\zeta(x,w)^{1-\psi} + (1-x)\rho^{\psi}\xi(x,w)^{1-\psi} \\
\frac{p_w}{p} &= \frac{\alpha_w}{\alpha} - (1-\psi)\frac{p}{\alpha}\left[x\rho^{\psi}\zeta^{1-\psi}\frac{\zeta_w}{\zeta} + (1-x)\rho^{\psi}\xi^{1-\psi}\frac{\xi_w}{\xi}\right] \\
\frac{q_x}{q} &= \frac{\alpha_x}{\alpha} - (1-\psi)\frac{p}{\alpha}\left[x\rho^{\psi}\zeta^{1-\psi}\frac{\zeta_x}{\zeta} + (1-x)\rho^{\psi}\xi^{1-\psi}\frac{\xi_x}{\xi} + \frac{\rho^{\psi}\zeta^{1-\psi}-\rho^{\psi}\xi^{1-\psi}}{1-\psi}\right] \\
\frac{q_{xx}}{q} &= \frac{\alpha_{xx}}{\alpha} - 2\frac{\alpha_x}{\alpha}\frac{q_x}{q} + 2\left(\frac{q_x}{q}\right)^2 - (1-\psi)\frac{q}{\alpha}\left[x\rho^{\psi}\zeta^{1-\psi}\left(\frac{\zeta_{xx}}{\zeta} + \frac{2\zeta_x}{x\zeta} - \psi\left(\frac{\zeta_x}{\zeta}\right)^2\right) + (1-x)\rho^{\psi}\xi^{1-\psi}\left(\frac{\xi_{xx}}{\xi} - \frac{2\xi_x}{(1-x)\xi} - \psi\left(\frac{\xi_x}{\xi}\right)^2\right)\right] \\
\frac{q_{ww}}{q} &= \frac{\alpha_{ww}}{\alpha} - 2\frac{\alpha_w}{\alpha}\frac{q_w}{q} + 2\left(\frac{q_w}{q}\right)^2 - (1-\psi)\frac{q}{\alpha}\left[x\rho^{\psi}\zeta^{1-\psi}\left(\frac{\zeta_{ww}}{\zeta} - \psi\left(\frac{\zeta_w}{\zeta}\right)^2\right) + (1-x)\rho^{\psi}\xi^{1-\psi}\left(\frac{\xi_{ww}}{\xi} - \psi\left(\frac{\xi_w}{\xi}\right)^2\right)\right] \\
\frac{q_{wx}}{q} &= \frac{\alpha_{wx}}{\alpha} - \frac{\alpha_w}{\alpha}\frac{q_x}{q} - \frac{\alpha_x}{\alpha}\frac{q_w}{q} + 2\frac{q_x}{q}\frac{q_w}{q} - (1-\psi)\frac{q}{\alpha}\left[x\rho^{\psi}\zeta^{1-\psi}\left(\frac{\zeta_{wx}}{\zeta} + \frac{\zeta_w}{x\zeta} - \psi\frac{\zeta_x}{\zeta}\frac{\zeta_w}{\zeta}\right) + (1-x)\rho^{\psi}\xi^{1-\psi}\left(\frac{\xi_{wx}}{\xi} - \frac{\xi_w}{(1-x)\xi} - \psi\frac{\xi_x}{\xi}\frac{\xi_w}{\xi}\right)\right]
\end{aligned}
$$

Drift: The drift of X_t is given by

$$\mu_{x,t} = x_t(1-x_t)\left[\left(\sigma_{b,t}-\sigma_{s,t}\right)\left(\eta_t - \omega_t(\sigma+\sigma_{q,t})\right) + \rho^{\psi_s}\zeta_t^{1-\psi_s} - \rho^{\psi_b}\zeta_t^{1-\psi_b}\right] - \kappa\left(x_t - \theta_b\right) \tag{A.13}$$

$$\mu_{w,t} = w_t\left[r_t + \sigma_{cb,t}\eta_t - \hat{T}_t - \left(\mu_{q,t} + g_t + \sigma\sigma_{p,t}\right)\right] - (1-w_t)(1-\omega_t^r)(\sigma+\sigma_{p,t})^2$$

Notice that $\mu_{x,t}$ can be computed given the diffusion terms derived above. Hence, we only need to solve for $\mu_{w,t}$. First, from (17) we obtain an expression for r_t:

$$r_t = \frac{A - \iota(g(q_t))}{q_t} - (\sigma+\sigma_{q,t})\eta_t + \mu_{q,t} + g_t + \sigma\sigma_{q,t} \tag{A.14}$$

Combining the previous two expressions, we get

$$\mu_{w,t} = (1-w_t)(1-\omega_t^r)(\sigma+\sigma_{q,t})(\eta_t - (\sigma+\sigma_{q,t})) - (1-w_t)(1-\omega_t^d)\frac{A-\iota(g(q_t))}{q_t} \tag{A.15}$$

using the fact $\hat{T}_t = \frac{1-\omega_t^d(1-w_t)}{w_t}\frac{A-\iota(g_t)}{q_t}$. □

A.5 Proof of proposition 5

Proof. Define the martingale G_t:

$$G_t = \int_0^t \frac{\pi_s}{\pi_0}e^{-\delta_b s}ds + \mathbb{E}_t\left[\int_t^\infty \frac{\pi_s}{\pi_0}e^{-\delta_b s}ds\right] = \int_0^t \frac{\pi_s}{\pi_0}e^{-\delta_b s}ds + e^{-\delta_b t}\frac{\pi_t}{\pi_0}p_t \tag{A.16}$$

where the second equality uses (33).

Computing the drift of the expression above and setting it to zero, we obtain the no-arbitrage condition:

$$\underbrace{\frac{1}{p_t} + \mu_{p,t} - \delta_b - r_t}_{\text{expected excess return}} = \sigma_{p,t}\eta_t \tag{A.17}$$

Applying Ito's lemma to $p_t = p(x_t, w_t)$, we obtain

$$\mu_{p,t} = \frac{p_{x,t}}{p_t}\mu_{x,t} + \frac{p_{w,t}}{p_t}\mu_{w,t} + \frac{\sigma_{x,t}^2}{2}\frac{p_{xx,t}}{p_t} + \sigma_{x,t}\sigma_{w,t}\frac{p_{xw,t}}{p_t} + \frac{\sigma_{w,t}^2}{2}\frac{p_{ww,t}}{p_t}$$

$$\sigma_{p,t} = \frac{p_{x,t}}{p_t}\sigma_{x,t} + \frac{p_{w,t}}{p_t}\sigma_{w,t}$$

Using the no-arbitrage condition, we obtain the PDE for the price of the bond:

$$0 = \frac{\sigma_{x,t}^2}{2}p_{xx,t} + \sigma_{x,t}\sigma_{w,t}p_{xw,t} + \frac{\sigma_{w,t}^2}{2}p_{ww,t} + (\mu_{x,t} - \sigma_{x,t}\eta,t)p_{x,t} + (\mu_{w,t} - \sigma_{w,t}\eta,t)p_{w,t} + 1 - (r_t + \delta_b)p_t \tag{A.18}$$

The average expected interest rate can be written as

$$r_t^e \equiv \delta_b^2 \mathbb{E}_t \left[\int_t^\infty \int_t^s e^{-\delta_b(s-t)} r_u du ds \right] = \delta_b \mathbb{E}_t \left[\int_t^\infty e^{-\delta_b(s-t)} r_s ds \right] \tag{A.19}$$

The expected path of interest rate can be compute by solving the following PDE:

$$0 = \frac{\sigma_{x,t}^2}{2} r_{xx,t}^e + \sigma_{x,t}\sigma_{w,t} r_{xw,t}^e + \frac{\sigma_{w,t}^2}{2} r_{ww,t}^e + \mu_{x,t} r_{x,t}^e + \mu_{w,t} r_{w,t}^e + \delta r_t - \delta r_t^e \tag{A.20}$$

Consider also the expected average interest rate up to period T:

$$r_t^e(x, w; T) \equiv \frac{\delta}{1 - e^{-\delta(T-t)}} \mathbb{E} \left[\int_t^T e^{-\delta(s-t)} r_s ds \right] \tag{A.21}$$

The expression above can be written as:

$$\begin{aligned}
r_t^e(x, w; T) &= \mathbb{E}_t \left[\int_t^{\Delta t} \frac{\delta e^{-\delta(s-t)}}{1 - e^{-\delta(T-t)}} r_s ds + e^{-\delta \Delta t} \frac{1 - e^{-\delta(T-t-\Delta t)}}{1 - e^{-\delta(T-t)}} \mathbb{E}_{t+\Delta t} \left[\int_{t+\Delta t}^T \frac{\delta e^{-\delta(s-t-\Delta t)}}{1 - e^{-\delta(T-t-\Delta t)}} r_s ds \right] \right] \\
&= \mathbb{E}_t \left[\int_t^{\Delta t} \frac{\delta e^{-\delta(s-t)}}{1 - e^{-\delta(T-t)}} r_s ds \right] + \frac{e^{-\delta \Delta t} - e^{-\delta(T-t)}}{1 - e^{-\delta(T-t)}} \mathbb{E}_t \left[r_{t+\Delta}^e(x_{t+\Delta t}, w_{t+\Delta t}; T) \right] \\
&= \mathbb{E}_t \left[\int_t^{\Delta t} \frac{\delta e^{-\delta(s-t)}}{1 - e^{-\delta(T-t)}} r_s ds \right] + \frac{e^{-\delta \Delta t} - e^{-\delta(T-t)}}{1 - e^{-\delta(T-t)}} \left[r_t^e(x_t, w_t; T) + \int_t^{t+\Delta t} \left(\dot{r}_s^e + \mathcal{D} r_s^e \right) ds \right] \\
&\approx \frac{\delta r_t \Delta t}{1 - e^{-\delta(T-t)}} + \frac{1 - \delta \Delta t - e^{-\delta(T-t)}}{1 - e^{-\delta(T-t)}} \left[r_t^e(x_t, w_t; T) + \left(\dot{r}_s^e + \mathcal{D} r_s^e \right) \Delta t \right]
\end{aligned}$$

where we denote a time derivative with a dot and $\mathcal{D} r_s^e$ denotes

$$\mathcal{D} r_s^e \equiv \mu_x r_x^e + \mu_w r_w^e + \frac{\sigma_x^2}{2} r_{xx}^e + \sigma_x \sigma_w r_{xw}^e + \frac{\sigma_w^2}{2} r_{ww}^e \tag{A.22}$$

Rearranging the expression above, we obtain

$$-\dot{r}_t^e = \frac{\delta(r_t - r_t^e)}{1 - e^{-\delta(T-t)}} + \mathcal{D} r_t^e \tag{A.23}$$

It is convenient to work $\tau = T - t$ instead of t directly. The expression above can be rewritten as

$$\frac{\partial r^e(x, w, \tau)}{\partial \tau} = \frac{\delta(r(x, w) - r^e(x, w, \tau))}{1 - e^{-\delta \tau}} + \mathcal{D} r^e(x, w, \tau) \tag{A.24}$$

subject to the boundary condition:

$$r^e(x, w, 0) = r(x, w) \tag{A.25}$$

Taking the limit as τ goes to zero, we have

$$\frac{\partial r^e(x, w, 0)}{\partial \tau} = \frac{1}{2} \mathcal{D} r^e(x, w, 0) \tag{A.26}$$

where I used the fact

$$\lim_{\tau \to 0} \frac{\delta(r(x,w) - r^e(x,w,\tau))}{1 - e^{-\delta\tau}} = -\frac{\partial r^e(x,w,0)}{\partial \tau} \tag{A.27}$$

□

B Numerical Solution

The computation of equilibrium is reduced to the solution of a system of partial differential equations (PDEs) involving $(\zeta(x,w), \xi(x,w))$. The following procedure shows how to obtain a pair of conditions involving (ζ, ξ) and its derivatives:

i. Compute $q(x,w)$ using the condition:

$$x\rho^{\psi}\zeta(x,w)^{1-\psi} + (1-x)\rho^{\psi}\xi(x,w)^{1-\psi} = \omega^d(x,w)\frac{A - \iota(g(q(x,w)))}{q(x,w)}$$

and differentiate the condition above to obtain the derivatives of $q(x,w)$.

ii. Compute (σ_x, σ_w) using (A.12) and (A.10) in the appendix.

iii. Applying Ito's lemma, compute $(\sigma_{q,t}, \sigma_{\zeta,t}, \sigma_{\xi,t})$

$$\sigma_{q,t} = \frac{q_{x,t}}{q_t}\sigma_{x,t} + \frac{q_{w,t}}{q_t}\sigma_{w,t}; \qquad \sigma_{\zeta,t} = \frac{\zeta_{x,t}}{\zeta_t}\sigma_{x,t} + \frac{\zeta_{w,t}}{\zeta_t}\sigma_{w,t}; \qquad \sigma_{\xi,t} = \frac{\xi_{x,t}}{\xi_t}\sigma_{x,t} + \frac{\xi_{w,t}}{\xi_t}\sigma_{w,t}$$

iv. Compute η_t using (25) and $(\sigma_{b,t}, \sigma_{s,t})$ using (21) and the analogous condition for $\sigma_{s,t}$.

v. Compute $(\mu_{x,t}, \mu_{w,t})$ using conditions (A.13) and (A.15) in the appendix.

vi. Applying Ito's lemma, compute $(\mu_{q,t}, \mu_{\zeta,t}, \mu_{\xi,t})$

$$\begin{aligned}
\mu_{q,t} &= \frac{q_{x,t}}{q_t}\mu_{x,t} + \frac{q_{w,t}}{q_t}\mu_{w,t} + \frac{1}{2}\left[\frac{q_{xx,t}}{q_t}\sigma_{x,t}^2 + 2\frac{q_{xw,t}}{q_t}\sigma_{x,t}\sigma_{w,t} + \frac{q_{ww,t}}{q_t}\sigma_{w,t}^2\right]; \\
\mu_{\zeta,t} &= \frac{\zeta_{x,t}}{\zeta_t}\mu_{x,t} + \frac{\zeta_{w,t}}{\zeta_t}\mu_{w,t} + \frac{1}{2}\left[\frac{\zeta_{xx,t}}{\zeta_t}\sigma_{x,t}^2 + 2\frac{\zeta_{xw,t}}{\zeta_t}\sigma_{x,t}\sigma_{w,t} + \frac{\zeta_{ww,t}}{\zeta_t}\sigma_{w,t}^2\right]; \\
\mu_{\xi,t} &= \frac{\xi_{x,t}}{\xi_t}\mu_{x,t} + \frac{\xi_{w,t}}{\xi_t}\mu_{w,t} + \frac{1}{2}\left[\frac{\xi_{xx,t}}{\xi_t}\sigma_{x,t}^2 + 2\frac{\xi_{xw,t}}{\xi_t}\sigma_{x,t}\sigma_{w,t} + \frac{\xi_{ww,t}}{\xi_t}\sigma_{w,t}^2\right];
\end{aligned}$$

vii. Compute r_t using (17).

viii. Plug $(r_t, \eta_t, \sigma_{b,t}, \sigma_{\zeta,t})$ into (20), analogously for savers, to obtain the system of PDEs.

The boundary conditions for the PDEs can be obtained by the behavior of the diffusion for (x_t, w_t)

at the boundaries:[39]

$$\lim_{x\to 0} \sigma_{x,t} = 0,\ \forall w \in [0,1]; \qquad \lim_{x\to 1} \sigma_{x,t} = 0,\ \forall w \in [0,1];$$
$$\lim_{w\to 0} \sigma_{w,t} = 0,\ \forall x \in [0,1]; \qquad \lim_{w\to 1} \sigma_{w,t} = 0,\ \forall x \in [0,1];$$

The numerical solution is a finite-difference implementation of a *method of lines with false transient*. The method consists of introducing a "false" time dimension (or considering the finite horizon version of the problem) and discretizing the derivatives involving (x, w) using finite differences. The time dimension is kept continuous, so we convert the problem from a two-dimensional system of PDEs to a 2N-dimensional system of ODE (where N is the number of points in the grid for (x, w)). The system of ODEs is solved using MATLAB's ODE suite.

[39]See Schiesser (1996) for a discussion of boundary conditions involving order reduction of PDEs.